Basic Systems of Rational Norm-Curves.

By J. R. Conner.

I.

Introductory.

§ 1. *Basic Systems Defined.*

The *base*, or $(n+2)$-point in a space of n dimensions, derives its importance from the fact that it is the figure of the greatest number of independent points that is projectively equivalent to any other such set. The assumption of two ordered bases in two separate or superposed spaces gives a collineation between these spaces.

We shall use the symbol S_n in referring to a flat space of n dimensions, and shall designate the rational norm-curve in n dimensions by the symbol R^n. An R^n has n^2+2n-3 constants. It is $n-1$ conditions on a curve in n dimensions to pass through a given point, and hence R^n's on $n+2$ points have

$$n^2+2n-3-(n-1)(n+2)=n-1$$

degrees of freedom. We shall call a system of ∞^{n-1} R^n's on $n+2$ points in n dimensions a *basic system of rational norm-curves*, and individual curves of such a system, basic R^n's. There is a unique R^n on $n+3$ points, in general, and hence through the general point of the S_n under consideration passes one and only one basic R^n.

§ 2. *Contact of Basic R^n's with Spreads in S_n.*

The following theorem is one of which use will be made throughout this paper. It is inserted here as a matter of convenience.

* *Basic R^n's in S_n having p-point contact with an $(n-1)$-way spread,*

$$(\alpha x)^m = 0,$$

of order m, touch at points of an $(n-p)$-way spread of order

$$m(m+1)(m+2)\ldots\ldots(m+p-1),$$

which is the complete intersection with α of $(n-1)$-way spreads of orders

$$m+1,\quad m+2,\quad \ldots\ldots,\quad m+p-1.$$

Proof: Take as base the $n+1$ points of the reference figure and the unit point. An R^n on these points and a point x is in general uniquely determined, and may be given parametrically by

$$\rho x_i = \frac{x_i}{1 - x_i t} \quad (i = 1, \ldots\ldots, n+1). \tag{1}$$

$t = \frac{1}{x_i}$ gives the points of reference, $t = \infty$ gives the unit point, and $t = 0$ gives the point x.

(1) may also be written

$$\sigma x_i = \frac{x_i \prod_{j=1}^{n+1} (1 - x_j t)}{1 - x_i t},$$

or

$$\sigma x_i = x_i \left[1 - s_{1,i} t + s_{2,i} t^2 - s_{3,i} t^3 + \ldots\ldots + (-)^n s_{n,i} t^n\right], \tag{2}$$

where $s_{j,i}$ is the sum of products j at a time of all the x's but x_i. The R^n (2) meets $(\alpha x)^m = 0$ in the mn points given by

$$[(\alpha x) - (\alpha x s_1) t + (\alpha x s_2) t^2 - \ldots\ldots]^m = 0, \tag{3}$$

where

$$(\alpha x) \equiv \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots,$$
$$(\alpha x s_j) \equiv \alpha_1 x_1 s_{j,1} + \alpha_2 x_2 s_{j,2} + \ldots\ldots.$$

* Compare Humbert: "Sur un Complexe Remarquable de Coniques," *Journal de l'École Polytechnique*, Cah. 64 (1894), p. 125. Humbert's argument that basic R^3's touch an n-ic surface at points of a curve of order $2n^2$ and osculate in $6n^3$ points is obviously incorrect. His equations (6) should be

$$f(x, y, z, t) = 0,$$

$$x^2 \frac{\partial f}{\partial x} + y^2 \frac{\partial f}{\partial y} + z^2 \frac{\partial f}{\partial z} + t^2 \frac{\partial f}{\partial t} = 0,$$

which is the form given further on.

Expanding (3) and arranging in ascending powers of t, we have

$$\left.\begin{aligned}&(\alpha x)^m - m(\alpha x)^{m-1}(\alpha x s_1)\,t + \left[\tbinom{m}{2}(\alpha x)^{m-2}(\alpha x s_1)^2 + m(\alpha x)^{m-1}(\alpha x s_2)\right]t^2\\ &\quad -\left[\tbinom{m}{3}(\alpha x)^{m-3}(\alpha x s_1)^3 + 2\tbinom{m}{2}(\alpha x)^{m-2}(\alpha x s_1)(\alpha x s_2) + m(\alpha x)^{m-1}(\alpha x s_3)\right]t^3\\ &\quad +\left[\tbinom{m}{4}(\alpha x)^{m-4}(\alpha x s_1)^4 + 3\tbinom{m}{3}(\alpha x)^{m-3}(\alpha x s_1)^2(\alpha x s_2) + \tbinom{m}{2}(\alpha x)^{m-2}(\alpha x s_2)^2\right.\\ &\quad \left.+ 2\tbinom{m}{2}(\alpha x)^{m-2}(\alpha x s_1)(\alpha x s_3) + m(\alpha x)^{m-1}(\alpha x s_4)\right]t^4 - \ldots\ldots = 0.\end{aligned}\right\}\quad(4)$$

If x is a point of $(\alpha x)^m = 0$, t is a factor of (4), and the remaining roots of (4) are the parameters of points of intersection of (2) with α. If we desire the basic R^n on x to touch the spread α at x, t^2 must be a factor of (4) and we must have

$$(\alpha x)^m = 0,$$
$$(\alpha x)^{m-1}(\alpha x s_1) = 0.$$

If we desire p-point contact, t^p must be a factor of (4); the coefficients of $1, t, t^2, \ldots, t^{p-1}$ in (4) must vanish. But these represent $(n-1)$-way spreads of orders

$$m,\quad m+1,\quad m+2,\quad \ldots,\quad m+p-1,$$

respectively. Hence the theorem.

The coefficients of (4) may be modified by the identities

$$\left.\begin{aligned}&(\alpha x s_1) \equiv (\alpha x)\sigma_1 - (\alpha x^2),\\ &(\alpha x s_2) \equiv (\alpha x)\sigma_2 - (\alpha x^2)\sigma_1 + (\alpha x^3),\\ &(\alpha x s_3) \equiv (\alpha x)\sigma_3 - (\alpha x^2)\sigma_2 + (\alpha x^3)\sigma_1 - (\alpha x^4),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}\right\}\quad(5)$$

where the σ's are symmetric functions of all the x's, and

$$(\alpha x^j) \equiv \alpha_1 x_1^j + \alpha_2 x_2^j + \ldots\ldots$$

Using the identities (5), we have from (4):

The conditions for p-point contact of a basic R^n with $(\alpha x)^m = 0$ at the point x are

$$\left.\begin{aligned}&(\alpha x)^m = 0,\\ &(\alpha x)^{m-1}(\alpha x^2) = 0,\\ &(m-1)(\alpha x)^{m-2}(\alpha x^2)^2 + 2(\alpha x)^{m-1}(\alpha x^3) = 0,\\ &(m-1)(m-2)(\alpha x)^{m-3}(\alpha x^2)^3 + 6(m-1)(\alpha x)^{m-2}(\alpha x^2)(\alpha x^3)\\ &\qquad\qquad + 6(\alpha x)^{m-1}(\alpha x^4) = 0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\quad(6)$$

The forms (6) may be obtained more directly by substitution of the coördinates of x as given by (1) in $(\alpha x)^m$, differentiating $p-1$ times as to t and putting $t = 0$

in the resulting expressions.* The spreads (6) behave very much like the successive polar spreads of α, and the modifications of the theorem for the case where α has multiple spreads are obvious. A specially interesting case arises when basic R^n's cannot have p-point contact with α without lying entirely on it; we shall see several applications of the theorem in this connection.

We have from (6):

Basic R^n's have p-point contact with an S_{n-1}, $(\alpha x) = 0$, where

$$(\alpha x) = (\alpha x^2) = (\alpha x^3) = \dots = (\alpha x^p) = 0. \tag{7}$$

II.

BASIC NORM-CURVES IN TWO DIMENSIONS.

§ 3. *Applications of the Theorem of Part I.*

The base in a plane is a set of four points. The basic system of norm-curves is here the pencil of conics on the four points. These conics define an involution on every line of the plane. The double points of this involution are the points of contact of the two basic conics that touch the line. The points of contact of basic conics with a given line, $(\alpha x) = 0$, are cut out by

$$(\alpha x^2) = 0.$$

Basic conics touch an m-ic curve

$$\alpha \equiv (\alpha x)^m = 0 \tag{1}$$

in the $m(m+1)$ points cut out by

$$(\alpha x)^{m-1}(\alpha x^2) = 0. \tag{2}$$

If (1) is on a base-point, (2) touches (1) there. If (1) is on a point of the diagonal triangle of the base, (2) passes through that point, but does not in general touch (1) there. More generally, if (1) has an i-fold point at a base-point, (2) has an i-fold point there, with the same tangents. If (1) has an i-fold point at a point of the diagonal triangle of the base, (2) has also an i-fold point there, but not with the same tangents in general.

(2) vanishes if the conditions for a node of α are satisfied; that is, if

$$(\alpha x)^{m-1}\alpha_i = 0 \quad (i = 1, 2, 3).$$

Hence (2) passes through all the nodes of α, and two is to be deducted from the number of actual contacts for each of the nodes.

*Humbert, *loc. cit.*

If α have a cusp at x, we must have

$$\frac{\partial \alpha}{\partial x_1}=\frac{\partial \alpha}{\partial x_2}=\frac{\partial \alpha}{\partial x_3}=0,$$

and

$$y_1^2\frac{\partial^2 \alpha}{\partial x_1^2}+y_2^2\frac{\partial^2 \alpha}{\partial x_2^2}+y_3^2\frac{\partial^2 \alpha}{\partial x_3^2}+2y_2y_3\frac{\partial^2 \alpha}{\partial x_2\partial x_3}+\ldots.$$

is a perfect square. This last expression, if a perfect square, is the square of

$$y_1\sqrt{\frac{\partial^2 \alpha}{\partial x_1^2}}+y_2\sqrt{\frac{\partial^2 \alpha}{\partial x_2^2}}+y_3\sqrt{\frac{\partial^2 \alpha}{\partial x_3^2}}=0, \tag{3}$$

the square roots being those definite roots determined by the conditions

$$\sqrt{\frac{\partial^2 \alpha}{\partial x_i^2}}\sqrt{\frac{\partial^2 \alpha}{\partial x_j^2}}=\frac{\partial^2 \alpha}{\partial x_i\partial x_j}. \tag{4}$$

(3) equated to zero is the cusp tangent to α at x. Now the tangent to (2) at x is, dropping the terms in $\frac{\partial \alpha}{\partial x_i}$,

$$\left.\begin{aligned} &y_1\left(x_1^2\frac{\partial^2 \alpha}{\partial x_1^2}+x_2^2\frac{\partial^2 \alpha}{\partial x_1\partial x_2}+x_3^2\frac{\partial^2 \alpha}{\partial x_1\partial x_3}\right)\\ &\quad+y_2\left(x_1^2\frac{\partial^2 \alpha}{\partial x_1\partial x_2}+x_2^2\frac{\partial^2 \alpha}{\partial x_2^2}+x_3^2\frac{\partial^2 \alpha}{\partial x_2\partial x_3}\right)\\ &\quad+y_3\left(x_1^2\frac{\partial^2 \alpha}{\partial x_1\partial x_3}+x_2^2\frac{\partial^2 \alpha}{\partial x_2\partial x_3}+x_3^2\frac{\partial^2 \alpha}{\partial x_3^2}\right)=0. \end{aligned}\right\} \tag{5}$$

(5), on account of the conditions (4), may be written

$$\left(y_1\sqrt{\frac{\partial^2 \alpha}{\partial x_1^2}}+y_2\sqrt{\frac{\partial^2 \alpha}{\partial x_2^2}}+y_3\sqrt{\frac{\partial^2 \alpha}{\partial x_3^2}}\right)\left(x_1^2\sqrt{\frac{\partial^2 \alpha}{\partial x_1^2}}+x_2^2\sqrt{\frac{\partial^2 \alpha}{\partial x_2^2}}+x_3^2\sqrt{\frac{\partial^2 \alpha}{\partial x_3^2}}\right)=0.$$

Hence (2) passes through all cusps of (1) and touches the cusp-tangents at the cusps. We have, then,

If an m-ic curve, not related to the base, have δ nodes and $\varkappa$ cusps, the number of basic conics having proper contact with it is

$$N=m(m+1)-2\delta-3\varkappa.$$

We can now find the order of the curve of contact of basic conics with a pencil of m-ics. An m-ic of the pencil can meet the curve only at its points of contact with basic conics, since there is only one m-ic of the pencil on a general

point of the plane, and at the base-points of the pencil of m-ics. We have, then, as the number of intersections of an m-ic with the curve,

$$m^2 + m(m+1) = m(2m+1).$$

This gives the theorem:

* *The curve of contact of a pencil of conics and a pencil of m-ics is a curve of order* $2m+1$.

If we have a pencil of m-ics

$$(\alpha x)^m + \lambda (\beta x)^m = 0,$$

the contacts of basic conics with the curves of this pencil are cut out by the corresponding curves of the pencil

$$(\alpha x)^{m-1}(\alpha x^2) + \lambda (\beta x)^{m-1}(\beta x^2) = 0.$$

Eliminating λ between these two equations, we have

$$\begin{vmatrix} (\alpha x)^m & (\beta x)^m \\ (\alpha x)^{m-1}(\alpha x^2) & (\beta x)^{m-1}(\beta x^2) \end{vmatrix} = 0 \qquad (6)$$

as the equation of the curve of contact. If $(\alpha x)^m$ have a node at a point y, $(\alpha x)^{m-1}(\alpha x^2)$ passes through y, and y is a simple point of (6). Similarly, if $(\alpha x)^m$ have a p-fold point at y, y is a $(p-1)$-fold point of (6). (6) obviously passes through the base-points of the pencil of m-ics. It is also on the four points of the base, and the three points of their diagonal triangle.

§ 4. *A Rational Sextic Associated with the Base.*

Certain curves that arise in connection with systems of basic conics in a plane are of importance in the discussion of higher dimensional cases. We mention some of these here, though the discussion of the special cases of these curves that we shall meet later will be easier in the light of facts that have not yet been brought out.

A sextic curve, Σ, is determined by the basic conics and a general conic, C, in the following manner: Through a point of intersection of a basic conic with C draw the tangent to C. *The locus of the point where this tangent meets the basic conic again is a rational sextic curve.*

For it obviously has nodes at each base-point, the double point being determined by the two basic conics on the two points of tangency of tangents from the base-point to C. Further, a basic conic, apart from intersections at

* This is a special case of a more general theorem. See Pascal: "Rep. der Höheren Math.," II, p. 432.

the base-points, can meet the curve only in the four points determined by its intersections with C, since there is one and only one basic conic through a point of the curve, and this must be its defining conic. We see, then, that a basic conic has $4 + 2 . 4 = 12$ intersections with Σ, and hence it is of order 6, being met by a conic in twelve points. Σ is in one-one correspondence with C and hence must be rational. It must touch C at the six points of contact of basic conics with C. For these basic conics obviously touch Σ there.

Four of the ten nodes of Σ are at the base-points. Since there is only one basic conic on a given point of the plane other than a base-point, the only way in which additional nodes can arise is for the pole as to C of a common chord of C and a basic conic ϕ to be on ϕ. If this happens for a given common chord of C and ϕ it will happen for the opposite common chord. Hence:

Six nodes of Σ lie by twos on three basic conics. The other four nodes are at the base-points.

A rational sextic in a plane has seventeen constants. Σ, being determined by a conic and a base, has $5 + 8 = 13$ constants. Three conditions on Σ are accounted for by the theorem as to the nodes. The additional condition seems to be the existence of a *perspective conic*, C.*

§ 5. *A Point-Sextic and a Line-Quartic Determined by Two Bases.*

The base in a plane determines a quadratic Cremona involution x, y such that x and y are apolar to all basic conics. The polar lines of x with regard to basic conics are on y. x and y are the points of contact of basic conics with the line xy. The base-points are fixed points of this transformation and the three points of the diagonal triangle are singular points. A line is carried by the transformation into a conic on the singular points. A conic goes into a rational quartic with nodes at the singular points.

There are four pairs of corresponding points on every conic of the plane,

* Stahl: "Zur Erzeugung der ebenen rat. Curven," *Math. Ann.*, XXXVIII. Two rational curves, one given in lines and the other in points, are in *perspective position* when they are in one-one correspondence and corresponding lines and points are incident. This requires, in general, that the two curves shall touch, or that the point-curve shall have a cusp, at every common corresponding point. For, let us say that the line-curve Λ meets the point-curve L at a common corresponding point a. Consider L as generated by a point of a line of Λ, suitably fixed. This point approaches a as the point of tangency of the tangent to Λ approaches a. But a variable point on a tangent to a curve, at an ordinary point and in the neighborhood of the point of tangency, cannot cross the curve. Hence our statement. It is an easy inference from Stahl's theory that a point-curve and a line-curve in perspective position must, in general, have $m + n - 2$ common corresponding points, m and n being the order and class respectively of the two curves.

namely, the eight points in which the conic meets its correspondent quartic. Assuming the basic conics as given, a pencil of conics Φ determines a curve of points x whose correspondent, y, as to the basic conics, lies on the same conic Φ with x. x is subject to the condition that the points x, y, determined by the basic conics, shall be apolar to the points x', y', the points of contact of conics Φ with the line xy. The locus of x, S say, is evidently on the base-points of Φ, since the involution x', y' is parabolic at these base-points. If x is a base-point of Φ, the point y determined by x will, then, satisfy the condition of apolarity to $x'y'$. A conic Φ meets S in $8+4=12$ points. Hence:

The locus of a point, x, whose correspondent, y, in the Cremona involution determined by the basic conics, lies on a conic with x and four fixed points of the plane, is a sextic curve S.

This sextic is invariant under the quadratic involution, corresponding points on it being interchanged. It follows that it must have nodes at the diagonal points of the base.

These facts may by reached analytically. Let us assume as base the points $(1, \pm 1, \pm 1)$ with the reference points as diagonal points. The Cremona involution determined by basic conics is now simply

$$\rho\, y_i = \frac{1}{x_i}. \tag{7}$$

Suppose the pencil of conics, Φ, to be

$$\Phi \equiv (\alpha\, x)^2 + \lambda\, (\beta\, x)^2 = 0. \tag{8}$$

The points x whose corresponding points lie on the same conic Φ with x must satisfy in addition, according to (7),

$$(\alpha/x)^2 + \lambda\, (\beta/x)^2 = 0, \tag{9}$$

where

$$(\alpha/x) \equiv \alpha_1/x_1 + \alpha_2/x_2 + \alpha_3/x_3.$$

Eliminating λ between (8) and (9), we have

$$S \equiv \begin{vmatrix} (\alpha\, x)^2 & (\beta\, x)^2 \\ (\alpha/x)^2 & (\beta/x)^2 \end{vmatrix} = 0, \tag{10}$$

as the equation of our sextic. $(\alpha/x)^2$ and $(\beta/x)^2$ are rational quartics with nodes at the diagonal points of the base. The form of (10) gives us with regard to the sextic S:

1) S is invariant under the involution (7).
2) S has nodes at the diagonal points of the base.
3) S is on the base-points of both pencils of conics.

The locus of points of contact of basic conics with the lines of a pencil is a curve of order $2.1+1=3$, as is otherwise known. This cubic is on the four base-points and the three diagonal points. It is evidently invariant under (7), points of contact on each line being interchanged. Every point of the plane determines a cubic of this sort; there are ∞^2 of them and they form a linear system. A cubic determined in this way by a point a meets S in $18-3.2-4=8$ points, apart from the seven. This says that through any point a of the plane there are four lines joining pairs of corresponding points on the sextic S. Hence:

Lines joining corresponding points of the sextic S touch a curve of class four.

The equation of this quartic may be written down at once by means of the Clebsch translation-principle. Given two pencils of binary quadratics

$$(\alpha x)^2+\lambda(\beta x)^2=0,$$
$$(\gamma x)^2+\mu(\delta x)^2=0,$$

the apolarity-condition of the Jacobians of the two pencils is

$$|\alpha\beta|\;|\gamma\delta|\;[|\alpha\gamma|\;|\beta\delta|+|\alpha\delta|\;|\beta\gamma|]=0,$$

or

$$|\alpha\gamma|^2\,|\beta\delta|^2-|\alpha\delta|^2\,|\beta\gamma|^2=0, \tag{11}$$

since

$$|\alpha\beta|\;|\gamma\delta|\equiv|\alpha\gamma|\;|\beta\delta|-|\alpha\delta|\;|\beta\gamma|.$$

The equation of our quartic is, from (11),

$$|\alpha\gamma\xi|^2\,|\beta\delta\xi|^2-|\alpha\delta\xi|^2\,|\beta\gamma\xi|^2=0, \tag{12}$$

this being the equation of the quartic determined by the two pencils of conics

$$(\alpha x)^2+\lambda(\beta x)^2=0,$$
$$(\gamma x)^2+\mu(\delta x)^2=0.$$

If $(\gamma x)^2+\mu(\delta x)^2$ is the pencil of conics on the base $(1, \pm 1, \pm 1)$, we have

$$\gamma_{ij}=\delta_{ij}=\Sigma\gamma_{ii}=\Sigma\delta_{ii}=0,$$

and (12) becomes,

$$\left.\begin{aligned}&\Sigma(\alpha_{22}\beta_{33}-\alpha_{33}\beta_{22})\,\xi_1^4+2\Sigma(\alpha_{11}\beta_{23}-\alpha_{23}\beta_{11})\,\xi_2\xi_3(\xi_2^2-\xi_3^2)\\&\quad+\Sigma\,[\alpha_{11}(\beta_{33}-\beta_{22})-\beta_{11}(\alpha_{33}-\alpha_{22})]\,\xi_2^2\xi_3^2\\&\quad+2\Sigma\,[\alpha_{23}(\beta_{33}-\beta_{22})-\beta_{23}(\alpha_{33}-\alpha_{22})+2(\alpha_{12}\beta_{13}-\beta_{12}\alpha_{13})]\,\xi_1^2\xi_2\xi_3=0.\end{aligned}\right\} \tag{13}$$

The terms not given may be obtained by cyclic permutation of the subscripts.

The equation (13) shows that the quartic touches the six lines $(1, \pm 1, 0)$, $(1, 0, \pm 1)$, $(0, 1, \pm 1)$, the six lines joining the base-points two and two. By symmetry we infer that it also touches the six lines joining the base-point of Φ.

It will be observed that two sextics are determined by the basic conics and the pencil Φ, one, S, whose equation we have found above, and another, S', the locus of pairs of corresponding points x', y' which lie on basic conics. The equation of S' may be obtained as we found that of S. It is, in determinant form,

$$S' \equiv \begin{vmatrix} 1, & 1, & 1 \\ x_1^2, & x_2^2, & x_3^2 \\ y_1^2, & y_2^2, & y_3^2 \end{vmatrix} = 0, \tag{14}$$

where

$$y_i = \frac{\partial}{\partial a_i} |a \alpha \beta| (\alpha x)(\beta x), \tag{15}$$

(15) being the Cremona involution determined by the conics Φ. The quartic (13) is symmetrically related to both pencils of conics and to both sextics. Its lines meet the two sextics in harmonic pairs.

III.

BASIC NORM-CURVES IN THREE DIMENSIONS.

§ 6. *Introductory.*

The base in three dimensions is a set of five points. There is a system of ∞^2 norm-curves (R^3's) on the five points, since an R^3 has twelve coördinates, and it is ten conditions on a curve in space to pass through five given points. In addition we have a linear system of ∞^4 quadric surfaces on the five base-points, which we shall call *basic quadrics*. In our discussion of the subject we shall make use of the following known theorems:

a) *Basic quadrics meet any plane α in conics apolar to a definite conic C_α in α.**

b) *Basic R^3's meet α in sets of three points apolar to C_α.*†

These sets of three points are very important in the consideration of the apparatus determined by basic R^3's on a plane. We shall use the symbol $\iota_3^{(3)}$ in referring to such a set of points. In general, we shall use the symbol $\iota_r^{(n)}$ in referring to a set of r points of a basic R^n on any flat under consideration.

c) *Basic R^3's tangent to α touch at points of C_α.*‡

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* Compare Loria, *R. Istituto Lombardo*, April, 1884.

† Reye: "Geometrie der Lage," Part II, pp. 223, ff.

‡ Reye: *loc. cit.*

d) *Six basic R^3's osculate α, the points of osculation being on C_α.**

e) *Basic R^3's tangent to α meet α again in points of a rational sextic curve with nodes at the ten points of the Desargues configuration (configuration B) determined on α by the five base-points.**

f) *Basic elliptic quartics (E^4's) meet α in sets of four points orthic to C_α.*†

By a set of four points *orthic* to C_α we mean four points such that any conic on them taken in points is apolar to C_α taken in lines. Dualistically we have sets of four lines orthic to C_α.

C_α may be defined as the conic with regard to which the configuration B determined on α by the base is self-polar. We name the points of the base 1, 2, 3, 4, 5, the ten lines joining two of these five points 12, etc., and the ten planes joining three of them 123, etc., and name the points and lines of the configuration B on α after the lines and planes of the base by which they are determined.

§ 7. *The Point-Sets $\iota_3^{(3)}$.*

Consider the whole set of basic R^3's as projected from a base-point, say 5. This gives us a pencil of quadric cones with vertex at 5 and on the four lines 15, 25, 35, 45. On each of these cones there is a single infinity of basic R^3's, one R^3 through every point of the cone. A basic R^3 through a point a of one of these cones must lie entirely on the cone, since it has seven points in common with it, four at 1, 2, 3, 4, two at 5, and one at a.

This pencil of quadric cones meets α in a pencil of conics on 15, 25, 35, 45, all of which are apolar to C_α by theorem a). 15, 25, 35, 45 are an orthic 4-point as to C_α. A defining characteristic of such a set of points is that the pole of the join of any two is on the opposite line of the complete 4-point. A conic of this pencil, being apolar to C_α, must contain a single infinity of 3-points apolar to C_α. Since one point not a base-point and not on a base-line is sufficient to determine uniquely a basic R^3, and since one point of a conic of the pencil is sufficient to determine uniquely a 3-point apolar to C_α, we have the theorem:

3-points $\iota_3^{(3)}$ on α are 3-points inscribed in conics of the pencil 15, 25, 35, 45, *and apolar to C_α, and conversely.*

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* Sturm, *Crelle's Journal*, Vol. LXXIX, p. 99. There are many points of contact between this paper of Sturm and the paper of Humbert cited earlier, on the one hand, and the present section of this paper on the other, but the point of view is somewhat different.

† This in an easy consequence of theorem a).

In the configuration B there are five orthic 4-points, namely:

15	25	35	45	(5)
14	24	34	54	(4)
13	23	43	53	(3)
12	32	42	52	(2)
21	31	41	51	(1)

These may be called the sets 5, 4, 3, 2, 1 respectively. Two sets i and j have one point and only one in common—the point ij. i-conics and j-conics can, then, meet in only three variable points, and these sets of points are always $\iota_3^{(3)}$'s. We have the theorem:

The three points of a set $\iota_3^{(3)}$ are on a conic with any orthic 4-point of B.

A set of points $\iota_3^{(3)}$ may be constructed when one of the three points is given by drawing the conic on 15, 25, 35, 45, and the given point. The other two points are the intersections with this conic of the polar line of the given point as to C_α. This gives us at once theorem c) above, and the following:

Basic R^3's osculate α at the six points of contact with C_α of conics on 15, 25, 35, 45.

*Conics of all orthic pencils of B touch C_α at the same six points.**

The sextic of theorem e) *is the locus of the point in which the tangent to C_α, at a point of a conic of a given orthic pencil of B, meets this latter conic again.*

§ 8. *The Sextic of Theorem* e).

Let us examine this sextic a little more closely. We saw in Part II that a sextic generated in this way has nodes at the base-points of the generating pencil, and touches C_α at the six points of contact of conics of that pencil. Since the pole of each line of an orthic 4-point is on the opposite line, and a line taken with its opposite line is a degenerate conic of the pencil, the six remaining nodes of the sextic are at the poles of the lines of our orthic 4-point, and we have theorem e).

Considering 15, 25, 35, 45 as a *base* in α, the equation of this sextic may be obtained by demanding that the polar line of a point x as to C_α touch the basic

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* This may be regarded as a geometrical determination of the five quartic involutions associated with a given sextic as Jacobian. Cf. Meyer, "Apolarität," pp. 305, ff. B is unique when the sextic is given.

conic on x. Choosing as reference points 15, 25, 35, with 45 as unit point, a basic conic may be written:

$$m_1 x_2 x_3 + m_2 x_3 x_1 + m_3 x_1 x_2 = 0, \tag{1}$$

with the condition

$$m_1 + m_2 + m_3 = 0. \tag{2}$$

C_a must be of the form

$$a_1 \xi_1^2 + a_2 \xi_2^2 + a_3 \xi_3^2 + 2\lambda(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) = 0, \tag{3}$$

the apolarity-condition of (1) and (3) being merely

$$\lambda(m_1 + m_2 + m_3),$$

which vanishes by (2). The point-equation of (3) is

$$(cx)^2 \equiv (a_2 a_3 - \lambda^2) x_1^2 + \dots + 2(\lambda^2 - \lambda a_1) x_2 x_3 + \dots = 0. \tag{4}$$

The basic conic on x is, parametrically,

$$y_i = \frac{x_i}{1 - x_i t} \qquad (i = 1, 2, 3). \tag{5}$$

The polar line of x as to $(cx)^2 \equiv C$ is

$$y_1 \frac{\partial C}{\partial x_1} + y_2 \frac{\partial C}{\partial x_2} + y_3 \frac{\partial C}{\partial x_3} = 0, \tag{6}$$

Substituting (5) in (6), we have the quadratic in t,

$$\sum^{123} \frac{\partial C}{\partial x_1} x_1 (1 - x_2 t)(1 - x_3 t) = 0,$$

which, arranged in powers of t, is

$$x_1 x_2 x_3 \Sigma \frac{\partial C}{\partial x_1} t^2 - \Sigma x_1 x_2 \left(\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2}\right) t + 2C = 0, \tag{7}$$

since

$$\Sigma x_1 \frac{\partial C}{\partial x_1} \equiv 2C.$$

The condition that (6) touch (5) is the discriminant of (7). The equation of the rational sextic is, then,

$$8 x_1 x_2 x_3 \Sigma \frac{\partial C}{\partial x_1} C - \left[\Sigma x_1 x_2 \left(\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2}\right)\right]^2 = 0.$$

This is of the form

$$8 x_1 x_2 x_3 L C - f_3^2 = 0, \tag{8}$$

where f_3 is a cubic, and L is the polar line of the unit point (45) as to C_a,

a configuration line. (8) obviously touches C_a at its intersections with f_3. Hence f_3 is a cubic on the six points of osculation of basic R^3's with α.

The four points 15, 25, 35, 45, and the conic C_a are sufficient to determine the configuration B on α. The points of B are the four base-points and the poles as to C_a of the six lines joining them. The lines of B are these six lines and the polar lines of the base-points. The coördinates of the points of B are:

$$\left.\begin{array}{llllll} 15 & (1,\ 0,\ 0), & 14 & (a_1,\ \lambda,\ \lambda), & 23 & [0,\ a_2-\lambda,\ -(a_3-\lambda)], \\ 25 & (0,\ 1,\ 0), & 24 & (\lambda,\ a_2,\ \lambda), & 31 & [-(a_1-\lambda),\ 0,\ a_3-\lambda], \\ 35 & (0,\ 0,\ 1), & 34 & (\lambda,\ \lambda,\ a_3), & 12 & [a_1-\lambda,\ -(a_2-\lambda),\ 0], \\ 45 & (1,\ 1,\ 1). & & & & \end{array}\right\} \quad (9)$$

We give in this table the names of the points in two-figure symbols.

(8) shows that the cubic f_3 meets the lines $x_1\, x_2\, x_3\, L$ at nodes of the sextic. This gives the theorem:

The six points of contact of basic conics with C, or the six points of osculation of basic R^3's with α, and the six points 15, 25, 35, 12, 23, 31 *are on a cubic curve.*

f_3 is, in full,

$$\tfrac{1}{2} f_3 \equiv \sum^{123} a_3 (a_2-\lambda)\, x_1^2 x_2 + [6\,\lambda^2 - 2\,\lambda\,(a_1+a_2+a_3)]\, x_1 x_2 x_3 = 0. \qquad (10)$$

(10) is, in terms of the cubic,

$$(c\,x)\,(c\,x^2) = 0,$$

which, as we have found, cuts out the contacts of basic conics with C_a,

$$\tfrac{1}{2} f_3 \equiv (x_1+x_2+x_3)\,C - (c\,x)\,(c\,x^2). \qquad (11)$$

The above theorem is immediately evident from (10) and (11) by actual substitution of the coördinates of the six points as given in the table (9). These six points are the vertices of the 4-line of B, 235, 135, 125, 123. This is the polar reciprocal as to C_a of the orthic 4-point 4, and is an orthic 4-line as to C_a.

There can be but one sextic with nodes at the ten points of a configuration B; two would have forty points of intersection. Although our method of generation is unsymmetrical, the sextic is symmetrical with regard to the configuration and may equally well be generated by any one of the other four orthic pencils of B. Again, the whole figure is self-dual, and any theorem referring to the configuration carries with it its reciprocal theorem; polar reciprocation in C_a is a sufficient proof for this statement. We have then:

The six vertices of any orthic 4-line of B and the six points of contact with C_a of conics of the range touching these lines, lie on a curve of order 3.

We have in this way five cubics. Any two of these cubics intersect only in the six points on C_a and in the three points of a configuration line. Also:

*The six lines of any orthic 4-point of B and the six lines of contact with C_a of conics on the 4 points, touch a curve of class 3.**

A metrically special, but projectively general, rational sextic with nodes at the ten points of a configuration B may easily be constructed by choosing 15, 25, as the circular points in the plane; 35, 45, are then inverse points in the director circle of C_a, and the pencil of conics on 15, 25, 35, 45 is then a pencil of circles on two points.†

§ 9. *Applications of the Theorem of Part I.*

From Part I we have, the four reference points and the unit point being taken as base:

Basic R^3's touch the surface

$$(\alpha x)^m = 0$$

at points of the curve of order $m(m+1)$ cut out of $(\alpha x)^m$ by the surface

$$(\alpha x)^{m-1}(\alpha x^2) = 0, \tag{12}$$

and osculate $(\alpha x)^m$ in the $m(m+1)(m+2)$ points of intersection of the three surfaces

$$\left.\begin{array}{c}(\alpha x)^m = 0, \\ (\alpha x)^{m-1}(\alpha x^2) = 0, \\ (m-1)(\alpha x)^{m-2}(\alpha x^2)^2 + 2(\alpha x)^{m-1}(\alpha x^2) = 0.\end{array}\right\} \tag{13}$$

(12) is on all nodes of $(\alpha x)^m$. Hence:

The locus of points of contact of basic R^3's with an m-ic surface passes twice through all nodes of the m-ic. In particular, if the m-ic have a double curve, this curve factors twice out of the curve of contact. A cuspidal curve of the m-ic factors three times out of the curve of contact.

The following is easily verified:

If $(\alpha x)^m$ passes through a base-point, (12) touches $(\alpha x)^m$ there, and the curve of contact has a node at the base-point.

§ 10. *The Basic R^3 Bisecant to a Given Line.*

The known theorem that there is one and only one basic R^3 which meets a given line π of space twice, is an immediate consequence of theorem b). If

* This cubic may be regarded as the locus of the lines of the complete 4-point of a set of one of the involutions of quartics on C_a.

† Salmon, "Conic Sections," p. 341, Ex. 2.

we choose any plane $\alpha+\lambda\beta$ of the pencil of planes containing π, the R^3 on the base-points and the pole of π as to the conic $C_{\alpha+\lambda\beta}$ in this plane must meet π twice, since the set of points $\iota_3^{(3)}$ determined by this pole on $\alpha+\lambda\beta$ is apolar to $C_{\alpha+\lambda\beta}$. The locus of this pole, as $\alpha+\lambda\beta$ turns around π, may be shown independently to be a basic R^3; we shall do this later. We shall call this basic R^3 the *bisecant* basic R^3 to π.

§ 11. *Cubic Surface Determined by a Line.*

As the plane $\alpha+\lambda\beta$ turns around π the conic $C_{\alpha+\lambda\beta}$ generates a cubic surface on the five base-points, and containing π. If a plane is on a base-point, the configuration B on that plane is merely a complete 4-line taken with the six lines projecting its six points from the base-point, and the conic C_α is in this case the lines ξ, ξ' through the base-point which are partners in the Cremona line-involution determined by the range of conics on the four lines. It follows that the planes joining π to each of the base-points are triple tangent planes of the cubic surface. The degenerate conics $C_{\alpha+\lambda\beta}$ on these planes form the ten lines of the surface that meet π. The conic C_α on a plane α is cut out of the plane by the quadric

$$(\alpha x^2)=0.^*$$

The conic $C_{\alpha+\lambda\beta}$ on a plane of the pencil

$$(\alpha x)+\lambda(\beta x)=0 \tag{14}$$

is cut out of the plane by the corresponding quadric

$$(\alpha x^2)+\lambda(\beta x^2)=0. \tag{15}$$

Eliminating λ from (14) and (15) we have as the locus of $C_{\alpha+\lambda\beta}$ the cubic surface

$$\begin{vmatrix} (\alpha x^2), & (\beta x^2) \\ (\alpha x), & (\beta x) \end{vmatrix} \equiv \Sigma\, \pi_{ij}\, x_i x_j (x_i - x_j)=0, \tag{16}$$

the cubic surface determined by the line

$$\pi_{ij}=\alpha_i\beta_j-\alpha_j\beta_i$$

and the base.

* Humbert (*loc. cit.*) gives this form of C_α. It may also be inferred from Reye, "Geometrie der Lage," II, p. 226.

The combinations $x_i x_j (x_i - x_j)$ in (16) are significant. The basic R^3 on x is

$$y_i = \frac{x_i}{1 - x_i t},$$

whence

$$\frac{dy_i}{dt} = \frac{x_i^2}{(1 - x_i t)^2},$$

$t = 0$ giving the point x. It follows that the basic R^3 on x passes through x with the tangent

$$p_{ij} = x_i x_j (x_i - x_j). \tag{17}$$

(16), then, appears as the condition that the tangent at x to the basic R^3 on x shall meet the line π. Hence:

The cubic surface (16) *is the locus of points x such that the tangent at x to the basic R^3 on x meets a given line π.*

If π is a tangent to a basic R^3, the cubic surface (16) has a node at the point of tangency, and the six lines through the node are the line π and the lines to the five base-points. Since an R^3 is projected from any point on it by a quadric cone, we have Cremona's theorem:

The six lines through the node of a nodal cubic surface are generators of a quadric cone.

§ 12. *Degeneration of Basic R^3's.*

It is of importance to point out the way in which basic R^3's may degenerate. Any base-line 12 taken with a conic on 3, 4, 5, and the point where 12 meets 345, is a degenerate basic R^3. A basic R^3 cannot meet 345, except at the points 3, 4, 5, without degenerating in this way. The section of the cubic surface (16) by 345 must, then, be the cubic curve of contact of conics of this pencil with lines of the pencil on the point where π meets 345.

§ 13. *A Ruled Cubic, and Its Connection with* (16).

If a point x of a basic R^3 run along a line p, the tangent at x to the variable R^3 generates a ruled cubic surface containing the line p as directrix.

For the ruled surface thus generated is met by any plane α on the line in the line itself and in the two tangents to basic R^3's at the points where the line meets the conic C_α. We indicate this ruled cubic surface determined by a line p by the symbol $[p]$.

The cubic surface (16) is general and contains twenty-seven lines. We have mentioned eleven of them, the line π and the ten lines meeting it.

Consider any one of the other sixteen, p say. All tangents to basic R^3's at points x of p meet π. π must therefore be contained in the ruled cubic surface $[p]$. Consider a section of $[p]$ by any plane α on π. The line π and the tangent to the basic R^3 on the point where p meets α are the only lines that can be included in this section. Hence the line π must count doubly. Again, a ruled cubic surface cannot contain more than two lines meeting all generators, and one of these must be the double line. If three skew lines meet all generators of a ruled surface the surface is a quadric. We have, therefore, the theorem:

The line π determining the cubic surface (16) *is the double line of any ruled cubic surface, $[p]$, determined by any line, p, of the surface skew to π.*

§ 14. *Complex of Tangents to Basic R^3's.*

We see from (17) that the tangents to all basic R^3's depend on the variation of four homogeneous parameters; that is, they lie in a complex. We proceed to determine its order. Suppose the p_{ij} to be the homogeneous coördinates of a point in five dimensions. The lines of space are represented by points of the quadric spread

$$Q \equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

(17), considered as a locus of points in five dimensions, is evidently a 3-way spread contained entirely in Q. The four planes x_i and the six $x_i - x_j$ are the ten planes joining the base-points. p_{ij} is the product of three planes on an edge of the tetrahedron of reference. All the p's, considered as cubic surfaces, contain the five base-points and the ten points in which a base-line meets the opposite plane. Hence

$$\Sigma\, \alpha_{ij}\, p_{ij} = 0 \tag{18}$$

is a cubic surface on these fifteen points. Now the order of a 3-way spread in five dimensions is the number of points in which it is met by an arbitrary plane, which may be represented analytically by three equations of the form (18). Three of the cubic surfaces (18) meet in $27 - 15 = 12$ variable points. Hence our complex is represented by a 3-way spread of order 12, and the complex must be of order 6. We have then:

*The totality of tangents to basic R^3's are in a sextic complex.**

(18) is a system of surfaces on fifteen points, but it is a 5-fold system, as we should expect. The fifteen points have the property that any cubic surface on

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* Sturm, *loc. cit.*

fourteen of them must be on the fifteenth, a fact easily verified by actually building up the equation of the cubic surface on fourteen of the points. (18) is merely the locus of points of tangency of basic R^3's with lines of the complex of tangents that lie in a given linear complex. (16) is obviously a special case, the linear complex being there the lines that meet a given line π.

The equation of the complex of tangents to basic R^3's may be obtained by elimination of the x's from (17), but it is more easily found indirectly. The form of equation (15) shows that π, the axis of the pencil of planes $\alpha + \lambda\beta$, is cut by $C_{\alpha+\lambda\beta}$ in pairs of an involution. This fact is also contained in theorem b), which shows that points of $C_{\alpha+\lambda\beta}$ on π are apolar to the two points of intersection with π of its bisecant basic R^3; hence the bisecant R^3 meets π in the double points of this involution. If the bisecant basic R^3 touches π, the two double points of the involution on π coincide, and the equation of the complex of tangents to basic R^3's may therefore be obtained by imposing the condition that this involution be parabolic. Represent the pencil of planes on π by $\alpha + \lambda\beta$, and the range of points by $a + \mu b$. The involution of points where the conics $C_{\alpha+\lambda\beta}$ meet π is cut out by

$$(\alpha x^2) + \lambda(\beta x^2) = 0.$$

The double points of this involution are given by the Jacobian of the quadratics in μ,

$$(\alpha x^2) = (\beta x^2) = 0,$$

when $a + \mu b$ is substituted for x. The discriminant of this Jacobian is

$$\begin{aligned}&[(\alpha a^2)(\beta b^2) - (\alpha b^2)(\beta a^2)]^2 \\ &\qquad - 4[(\alpha a^2)(\beta ab) - (\beta a^2)(\alpha ab)][(\alpha ab)(\beta b^2) - (\beta ab)(\alpha b^2)] = 0,\end{aligned}$$

and writing in this

$$\pi_{ij} = \alpha_i\beta_j - \alpha_j\beta_i = p_{kl}, \quad p_{ij} = a_i b_j - a_j b_i,$$

we have, as the equation of the complex,

$$\overset{6}{\Sigma} p_{12}^4 p_{34}^2 - 2\overset{12}{\Sigma} p_{21}^2 p_{23}^2 p_{41} p_{43} + 2\overset{3}{\Sigma} p_{12}^2 p_{34}^2 (p_{14} p_{32} + p_{13} p_{42}) = 0. \qquad (19)$$

§ 15. *A Theorem of Sturm and Its Use.*

There being a double infinity of basic R^3's, if a single condition is imposed on a basic R^3 the R^3 will lie on a surface. A theorem of Sturm is important in this connection:*

* Sturm, *loc. cit.*

Two loci of basic R^3's can meet only in basic R^3's and in parts of degenerate basic R^3's (in general, base-lines).

We give Sturm's proof of this, since it applies almost without change to higher cases. If x is a point of intersection of the two surfaces, the basic R^3 on x must lie on both surfaces, since this R^3 is uniquely determined by x, and since, each surface being a locus of basic R^3's, there must be a basic R^3 in each surface through the point x.

As an illustration of the practical application of this theorem, we prove that there are five basic R^3's meeting two lines π and π' — a fact which Sturm reached in another way. Basic R^3's meeting a line π lie on a surface whose order is obviously the number of basic R^3's meeting two lines. Now the surface determined by π contains each of the ten base-lines singly; the basic R^3 of which 12 is a part is 12 taken with the conic on 3, 4, 5 and the points where 12 and π meet 345. Assume now that the two surfaces π and π' meet in n basic R^3's. We have, then, that the total curve of intersection is a curve of order n^2, n being the order of each surface. In this intersection are the ten base-lines and n basic R^3's. This gives the equation

$$n^2 = 3n + 10 \quad \text{or} \quad n = 5.$$

§ 16. *The Locus of Basic R^3's Meeting a Line.*

We have, then, that basic R^3's which meet a given line π lie on a quintic surface. The bisecant basic R^3 to π is obviously a double curve of this surface. The surface contains the ten base-lines and has triple points at the five base-points. It may be pointed out in passing that the surface is rational, and may be considered as mapped from a plane by quartic curves on the ten points of intersection of five lines and on an eleventh point K.* The point K maps into the line π. The five lines in the plane map into the base-points, and their ten points of intersection into the ten base-lines. The two tangents from K to the conic on the five lines map into the basic R^3 bisecant to π, points of these two lines which map into the same point of the surface being determined by tangents to the conic.

This quintic surface is met by a general plane in a quintic curve with 3 nodes. It is met by a plane $\alpha + \lambda\beta$ on π in the line π and a quartic curve with a node at the pole, M, of π as to $C_{\alpha+\lambda\beta}$. It is clear from theorem b) that

* Compare Clebsch, *Math. Ann.*, I.

the joins of corresponding points on this quartic (points lying on the same basic R^3) are polar lines of points of π as to $C_{\alpha+\lambda\beta}$ and hence must pass through M. Since an R^3 is projected from any point of it by a quadric cone, we have:

This quartic is the locus of points on $\alpha+\lambda\beta$ *from which the five base-points and the point* M *are projected by six lines of a quadric cone.*

It is the section by $\alpha+\lambda\beta$ of the Weddle quartic surface with nodes at the six points $1, 2, 3, 4, 5, M$, the section of a Weddle quartic by a plane through a node. It meets π in the two points of $C_{\alpha+\lambda\beta}$ and in the two points of intersection with π of its bisecant basic R^3, these four points forming harmonic pairs. Any point P of space determines with regard to an R^3 a point P' such that P and P' are apolar to all quadrics on the R^3. PP' is a bisecant line of the R^3 and P and P' are apolar to the points of intersection of the R^3 with PP'. Since π may be considered as *any* bisecant line to the R^3 on $1, 2, 3, 4, 5, M$, we see from the above that the transformation PP' set up by the R^3 on its six nodes leaves a Weddle surface unaltered.*

The six points of tangency of tangents to the quartic in $\alpha+\lambda\beta$ are on $C_{\alpha+\lambda\beta}$. These are the six tangents through M to the complex curve of (19) on $\alpha+\lambda\beta$. The curve of this complex on any plane α is the polar reciprocal of the sextic of theorem e) in the conic C_α. The cone of rays of the complex through any point M is the enveloping cone from M to the Weddle surface determined by the five base-points and M. The points of contact of basic R^3's are points of contact with the surface.

We have made the statement that the locus of the pole of a line π as to the conic $C_{\alpha+\lambda\beta}$ on a plane $\alpha+\lambda\beta$ containing π, is a basic R^3 meeting π twice. The order of the curve may be found by considering its points of intersection with a plane $\alpha+\lambda\beta$. These points are the pole of π as to $C_{\alpha+\lambda\beta}$, and such points of the curve as may fall on π. For a point of the locus to be on π, π must touch a conic $C_{\alpha+\lambda\beta}$. But this happens twice, since these conics meet π in pairs of an involution. Hence the locus is an R^3 meeting π twice. It is a basic R^3, since the pole as to the degenerate conic on the plane joining π to a base-point is the node of this conic — the base-point itself.

§ 17. *Curves Determined on Surfaces by Basic R^3's.*

We have seen that basic R^3's touch an m-ic surface at points of a curve, F, of order $m(m+1)$, cut out of the m-ic surface by an $(m+1)$-ic surface. We

* Schoute, *Niew Archief*, (2), 4, (1899), p. 97.

ask for the order of the locus, G, of points in which basic R^3's tangent to an m-ic surface meet this surface again. This may be determined by finding in how many points G meets a base-plane. All points of intersection of G with a base-plane are determined by degenerate basic R^3's, since a basic R^3 cannot meet a base-plane except at a base-point without degenerating. Every base-plane meets the m-ic surface in an m-ic curve, and every base-line meets it in m points. There being $m(m+1)$ conics of a pencil that touch an m-ic curve we have m $m(m+1)$-fold points of G on every base-line, and since there are three base-lines on every base-plane, this gives $3m^2(m+1)$ intersections with G. In addition we have on a base-plane the points in which conics of the pencil which are tangent to the m-ic curve meet this curve again. There are $(2m-2)m(m+1)$ of these points. Hence the order of G is

$$3m^2(m+1)+(2m-2)m(m+1)=m(m+1)(5m-2).$$

We have:

Basic R^3's that touch a general m-ic surface meet the surface again in points of a curve, G, of order $m(m+1)(5m-2)$. G has $10m$ $m(m+1)$-fold points at the $10m$ points in which the base-lines meet the surface.

The sextic of theorem e) is a special case of this theorem.

G can meet F, or the $(m+1)$-ic surface cutting out F, in only two ways:

1) At a point of contact of a basic R^3 osculating the m-ic surface;

2) At a point of contact of a basic R^3 bitangent to the surface.

It touches F in the first case; merely intersects in the second. The number of points of intersection of the second kind is

$$m(m+1)^2(5m-2)-2m(m+1)(m+2)=m(m-1)(m+1)(5m-6).$$

Hence:

There are $\frac{1}{2}m(m^2-1)(5m-6)$ basic R^3's bitangent to a general m-ic surface.

The order of G may be obtained in another way. The locus of basic R^3's that meet a given line is a quintic surface containing the base-lines singly. We see from what we have said above that the base-lines are $m(m+1)$-fold lines on the locus of basic R^3's tangent to an m-ic surface. These two loci can meet only in basic R^3's and in base-lines. We have, then, if x is the order of the locus of tangent basic R^3's,

$$5x-10m(m+1)=3x \text{ or } x=5m(m+1).$$

Therefore:

Basic R^3's tangent to an m-ic surface lie on a surface of order $5m(m+1)$.

This surface touches the m-ic along F and cuts out the curve G. Hence the order of G is

$$5\,m^2(m+1) - 2\,m\,(m+1) = m\,(m+1)\,(5\,m - 2).$$

§ 18. *Ruled Surface Determined by a Curve.*

If a point x of a variable basic R^3 run along an m-ic curve, the tangent to this R^3 at x generates a ruled surface. The order of this surface is the number of generators met by an arbitrary line, π. Now the line π determines a cubic surface (16) which meets the m-ic curve in $3m$ points. The tangent at x to the R^3 on a point x of this surface meets π. Hence:

The locus of the tangent at x to the basic R^3 on a point x of an m-ic curve is a ruled surface of order $3m$.

Since the curve of contact of basic R^3's with an m-ic surface is of order $m\,(m+1)$, it follows that:

The tangents to basic R^3's tangent to an m-ic surface at points of contact with the surface lie on a ruled surface of order $3m\,(m+1)$.

If the m-ic surface is a quadric, this ruled surface of contact-tangents is of order 18. It meets the quadric in a curve of order 36. Out of this curve the curve F, of order 6, must factor twice, leaving a curve of order 24 to be accounted for. Now a tangent to a tangent R^3 touches the quadric and can meet in no further point without lying entirely on it. It follows that the curve of intersection of the quadric and ruled surface of contact-tangents is the curve F taken twice, and twenty-four lines. This is also obvious from another point of view. There must be twelve lines of each regulus on the quadric in the sextic complex of tangents to basic R^3's.

§ 19. *Degeneration of the Curve F. The Quintic Surface with Five Triple Points.*

If an m-ic surface contain base-lines or basic R^3's, these will appear in the curve F. For instance, a Weddle surface with nodes at the base-points contains the ten base-lines and the R^3 on the six nodes. The curve F is in this case the ten base-lines, the R^3 on the nodes, and a curve of order $4\,.\,5 - 10 - 3 = 7$. It is evident from what we have said that this septimic is the curve of tangency of the enveloping cone from the remaining node.

If the m-ic is a quadric on the base-points, a basic R^3 cannot touch the surface without lying entirely on it, as tangency requires seven points common

to the R^3 and the quadric. It is easy to show that there is actually a curve F in this case. A basic quadric is

$$\Sigma\, a_{ij}\, x_i\, x_j = 0,$$

where

$$\Sigma\, a_{ij} = 0.$$

F is cut out by the cubic surface

$$\Sigma\, a_{ij}\, x_i\, x_j\, (x_i + x_j) = 0,$$

which neither vanishes nor factors in general. The curve F can be nothing but basic R^3's. The order of F being 6, we have that *there are two basic R^3's on a basic quadric,* a theorem of Reye. Given a pencil of basic quadrics

$$\Sigma\, a_{ij}\, x_i\, x_j + \lambda\, \Sigma\, \beta_{ij}\, x_i\, x_j = 0, \tag{20}$$

with the conditions

$$\Sigma\, a_{ij} = \Sigma\, \beta_{ij} = 0,$$

the basic R^3's on quadrics of this pencil are cut out by

$$\Sigma\, a_{ij}\, x_i\, x_j\, (x_i + x_j) + \lambda\, \Sigma\, \beta_{ij}\, x_i\, x_j\, (x_i + x_j) = 0. \tag{21}$$

Eliminating λ from (20) and (21) we have as the locus of basic R^3's on quadrics of the pencil (20) the quintic surface

$$\begin{vmatrix} \Sigma\, a_{ij}\, x_i\, x_j\,, & \Sigma\, \beta_{ij}\, x_i\, x_j \\ \Sigma\, a_{ij}\, x_i\, x_j\, (x_i + x_j), & \Sigma\, \beta_{ij}\, x_i\, x_j\, (x_i + x_j) \end{vmatrix} = 0,$$

or

$$\left.\begin{aligned} &\overset{12}{\Sigma}\, (a_{12}\, \beta_{13} - a_{13}\, \beta_{12})\, x_1^2\, x_2\, x_3\, (x_2 - x_3) \\ &\qquad + \overset{6}{\Sigma}\, (a_{12}\, \beta_{34} - a_{34}\, \beta_{12})\, x_1\, x_2\, x_3\, x_4\, (x_1 + x_2 - x_3 - x_4) = 0. \end{aligned}\right\} \tag{22}$$

This is, as we shall see, *the general quintic surface with five triple points.* It must contain the base-lines singly, since it can meet the locus of basic R^3's which meet a line in only five basic R^3's, — the R^3's on the five points where the line meets the surface, — and hence the ten base-lines must be contained in it to make up the total intersection of order 25 of the two surfaces. A basic R^3 cannot meet (22) without lying entirely on it; hence the fifteen intersections of basic R^3's with the surface must all be at base-points. The base-points are therefore triple points of (22). A quadric of the generating pencil meets the surface in the two basic R^3's on the quadric, and in a basic E^4, the base-curve of the pencil. All of these facts might have been inferred from the equation of the surface.

The locus of the sextic curve of contact of basic R^3's with quadrics of the general pencil,

$$(\alpha x)^2 + \lambda (\beta x)^2 = 0,$$

is the quintic surface

$$\begin{vmatrix} (\alpha x)^2 & (\beta x)^2 \\ (\alpha x)(\alpha x^2) & (\beta x)(\beta x^2) \end{vmatrix} = 0. \tag{23}$$

(23) is on the base-points (singly) and contains the E^4 defined by the pencil. It passes through each of the base-points with the same tangent plane as the quadric of the pencil containing that point. A basic R^3 meets (23) in fifteen points; five are at base-points, and the other ten points are the ten points of tangency of quadrics of the pencil with the R^3. That there are just ten such points is evident from the fact that quadrics of the pencil cut out on the R^3 a binary sextic of the form

$$(\alpha t)^6 + \lambda (\beta t)^6 = 0,$$

and the discriminant of the binary sextic is of degree 10.

The locus of basic R^3's meeting a line is a special case of (22); in this case the pencil of quadrics contains the line and its bisecant basic R^3. We may obtain the equation of this surface in another form. A pencil of basic quadrics on the line joining the points a and b may be written:

$$\Sigma\, \alpha_{ij}\, x_i\, x_j = 0,$$

with the conditions

$$\begin{aligned} &\Sigma\, \alpha_{ij} = 0, \\ &\Sigma\, \alpha_{ij}\, a_i\, a_j = 0, \\ &\Sigma\, \alpha_{ij}\, b_i\, b_j = 0, \\ &\Sigma\, \alpha_{ij}\, (a_i\, b_j + a_j\, b_i) = 0. \end{aligned}$$

The two basic R^3's on a quadric of the pencil are cut out by

$$\Sigma\, \alpha_{ij}\, x_i\, x_j\, (x_i + x_j) = 0.$$

Eliminating α_{ij} from these six equations we have the quintic surface in the form of a 6-row determinant:

$$|\, x_i\, x_j, \quad x_i\, x_j\, (x_i + x_j), \quad 1, \quad a_i\, a_j, \quad b_i\, b_j, \quad a_i\, b_j + a_j\, b_i \,| = 0. \tag{24}$$

The argument which we have used to show that the order of the locus of basic R^3's tangent to an m-ic surface is $5m(m+1)$ may be used to show that the order of the locus of basic R^3's meeting an m-ic curve is $5m$. The same argument shows that the order of a locus of basic R^3's is completely determined

by the multiplicity of the base-lines in the surface. If the multiplicity of these lines is m, the order of the surface is $5m$. If the m-ic above is on the base-points, the number of points apart from base-points in which it meets the base-planes is reduced by three, the multiplicity of the base-lines in the surface obtained is reduced by three, and the order of the surface is reduced by fifteen. The order of the locus of basic R^3's meeting a basic E^4 is therefore $5 . 4 - 15 = 5$. Basic quadrics on the E^4 meet one of these R^3's in six points, and hence some quadric of the pencil must contain it entirely. An R^3 on a quadric meets an E^4 on the quadric in six points, the six points in which it meets the quadric cutting out the E^4. Hence the theorem:

The surface (22) *may be considered as the locus of basic R^3's meeting a basic E^4.*

The following theorem is also an easy inference from the argument of this section:

If an m-ic curve have p_i-fold points at the base-points i, the locus of basic R^3's meeting the m-ic is a surface of order $5m - 3\Sigma p_i$, having the line 12 *as $(m - p_3 - p_4 - p_5)$-fold line, etc.*

Whence:

The locus of basic R^3's meeting an R^3 on four of the five base-points is a 4-*nodal cubic surface with nodes at four of the base-points and on the remaining base-point.*

§ 20. *Plane Sections of the Quintic Surface with Five Triple Points.*

The section of the surface (22) by a plane α is a curve of interest. We approach this curve from another point of view. By theorem f) a basic E^4 meets any plane α in a set of four points orthic to C_α. Every conic of the pencil, Φ say, on these four points is apolar to C_α. These conics are the sections by α of basic quadrics on the basic E^4. There are two sets of three points $\iota_3^{(3)}$ on each conic of the pencil, since there are two basic R^3's on a basic quadric. The locus of sets $\iota_3^{(3)}$ on conics of the pencil Φ is met by one of these conics in the base-points of the pencil and in the six points of the two sets $\iota_3^{(3)}$,—ten points in all. This shows that the locus is a quintic curve, Q say. Q is on the ten points of the configuration B in α. For, consider the conic Φ on 12. This conic being apolar to C_α cuts 345, the polar line of 12 as to C_α, in a pair of points apolar to C_α. But 12 taken with any two points of 345 which are apolar to C_α is a set $\iota_3^{(3)}$. Hence:

Q is on the ten points of the configuration B in α. It meets a line of the configuration in the three configuration points and in two further points apolar to C_α.

It may be pointed out in passing that these two further points form a 2-point $\iota_2^{(3)}$, *as determined by the configuration B set up by the pencil Φ.*

Since the 3-points $\iota_3^{(3)}$ are apolar to C_a, we have the theorem:

The lines joining 3-points $\iota_3^{(3)}$ inscribed in conics of a pencil orthic to C_a touch a curve, K, of class 5.

For K is the polar reciprocal of the quintic Q in the conic C_a. This gives the further theorem:

Lines bisecant to basic R^3's which lie on quadrics of a basic pencil are in a quintic complex.

For there are ∞^3 of these lines; $\infty^1 R^3$'s and ∞^2 lines bisecant to an R^3. The lines enumerated in this way are all distinct, since there is only one basic R^3 bisecant to a given line. Hence they lie in a complex, and the fact that K is the complex-curve on the plane α gives us the above theorem.

For Q to have a node, in general, would require the surface (22) to have a double curve, which is not true. Hence:

The curves Q and K are quintics of genus 6.

If the pencil Φ is a line m taken with the pencil of lines on its pole, M, as to C_a, the quintic Q reduces to m taken with a quartic with node at M. This quartic, we have seen, is the section by α of the Weddle surface with nodes at 1, 2, 3, 4, 5, M. Similarly the curve K reduces to the point M and a line-quartic with m as a double line.

Since K is the polar reciprocal of Q as to C_a, we have:

K touches the polar lines of the base-points of Φ. It also touches the lines of the configuration B.

Calling the four base-points of Φ a, b, c, d, a degenerate conic $ab - cd$ must contain two 3-points $\iota_3^{(3)}$. Each line ab and cd must contain two points of a set, since every line has a unique bisecant basic R^3. Hence:

K touches the six lines joining the base-points of Φ.

It follows that:

Q passes through the six vertices of the 4-line which is the polar reciprocal as to C_a of the base 4-point of Φ.

We have mentioned the duality of figures associated with a configuration B. The 3-point $\iota_3^{(3)}$ and the 3-line on these points are dual figures. These sets of three lines are determined by an orthic 4-line of B in exactly the dual of the way in which sets $\iota_3^{(3)}$ are determined by an orthic 4-point. There are two points determined on every line of the plane, — the two points of intersection of its

bisecant basic R^3. Dually, two lines are determined on every point of the plane,—the two lines to the other points of the set $\iota_3^{(3)}$ determined by that point. These sets of points and lines are determined completely, and in dual ways, by the configuration B, without reference to the base in three dimensions. The two lines through a point a on α may be regarded as the two generators cut by α out of the quadric cone projecting 1, 2, 3, 4, 5 from a.

The relation between Q and K is, of course, completely dual. Q may be regarded as the locus of point-sets $\iota_3^{(3)}$ inscribed in conics of a pencil orthic to C_α, or as the locus of vertices of line-sets circumscribed about conics of a range orthic to C_α. Similarly K may be regarded as the locus of line-sets inscribed in conics of a pencil orthic to C_α, or as the locus of point-sets circumscribed about conics of an orthic range. The pencil and range are polar reciprocals. It is known that two 3-lines inscribed in a conic touch a conic, and dually. A conic of the pencil determines in this way a conic of the range, and *vice versa.* Incidentally we have the theorem:

If a conic (in lines) is apolar to C_α, there are just two line-sets $\iota_3^{(3)}$ circumscribing it. The conic on the vertices of these two 3-lines is apolar (in points) to C_α.

From what we have said we have:

The curve Q meets tangents to the curve K in pairs of points apolar to C_α, and dually.

The following theorem is also evident:

The bisecant basic R^3 meets any line on α in the Jacobian pair of the points in which the line meets C_α and the points in which it is touched by conics of an orthic pencil of B. For the points of the bisecant R^3 are cut out by a conic of an orthic pencil.

The base-points of Φ, a 4-point orthic to C_α, determine a second configuration, B', on α. We may state the facts found above in another way:

A second configuration B' with the same polarity C_α as B determines two quintic curves, a point-quintic Q and a line-quintic K, polar reciprocals as to C_α. Q contains the points of both configurations and cuts the lines of each configuration in a further pair of points apolar to C_α,—in fact, sets $\iota_2^{(3)}$ as determined by the other configuration.

It will be noticed that the two configurations B and B' bear exactly symmetrical relations to the quintics Q and K. The quintics Q and Q' determined by considering B or B' as fundamental meet the lines of the two configurations in the same points, sixty points, and must therefore coincide. Similarly B and B' play symmetrical rôles with regard to K.

§ 21. *Configurations on the Plane Section.*

Q, being entirely symmetrical with regard to both configurations, may equally well be generated by conics on any other orthic 4-point of the configuration B'. These two orthic 4-points determine two basic E^4's with a point in common, — the common point of the two 4-points. The first E^4 is on the surface (22). But the second must also lie on (22). For the new E^4 determines a quintic surface which meets α in the curve Q, since any two orthic 4-points of B' determine the same quintic curve Q. The two quintic surfaces have this curve in common, and hence must contain all basic R^3's meeting this curve; *i. e.*, they must coincide.

The fact that the new E^4 meets the old E^4 in a point on an arbitrary plane α indicates that there are an infinity of basic E^4's on the surface (22). We may see in another way that this is true. Any two basic R^3's lie on one and only one basic quadric. Choose any two basic R^3's on (22) that are not on a quadric of the generating pencil. These R^3's determine a basic quadric which meets (22) in the two R^3's and in a quartic curve, — *basic* since (22) has triple points at base-points. This quartic is a basic E^4. For an E^4 on a quadric cuts the generators of both systems twice; an R^4 on a quadric cuts the generators of one system three times and of the other once. In fact, the unique quadric on an R^4 is the locus of lines trisecant to the curve. Now any generator of the quadric above meets (22) in five points. It meets the pair of R^3's in three points,—the points in which it meets the cubic surface cutting them out. Hence the residual quartic meets every generator of both systems of the cutting quadric twice and must be an E^4. Hence:

The surface (22) *contains an infinity of basic E^4's.*

Again, any basic quadric on an E^4 of (22) cuts out the E^4 and a sextic with five actual nodes. A sextic in space with five nodes projects from any one of these nodes into a plane quartic with four nodes; *i. e.*, it is projected from any one of its nodes by a pair of quadric cones. The sextic must, then, break up into two R^3's on the five nodes. We see therefore that any quadric on a basic E^4 of (22) cuts out a pair of basic R^3's. There are ∞^2 quadrics on pairs of basic R^3's, and ∞^1 quadrics on each basic E^4. Hence:

The system of basic E^4's on (22) *is singly infinite.*

The following theorems are also obvious from what we have said:

(22) *is the locus of basic R^3's on quadrics of the pencil on any basic E^4 of the surface.*

Two E^4's of the system on (22) *have one and only one point in common, and there are two of these E^4's through every point of* (22).

On the plane α:

There are ∞^1 configurations B inscribed in Q and having the same polarity, C_a.

The locus of the lines of these configurations is the quintic K, the polar reciprocal of Q to C_a.

Any quintic surface with triple points at the five base-points is a locus of basic R^3's, since a basic R^3 meeting this surface in one point not a base-point has sixteen intersections with it and must lie on it. The argument used above applies throughout to such a surface, and we have:

The surface (22) *is the general quintic surface with five triple points.*

Let us apply the facts we have obtained to the special surface (24), the locus of basic R^3's meeting a line p. Basic quadrics meet the nodal curve of (24),—the bisecant R^3 to p,—in one point only apart from base-points. Basic quadrics on pairs of basic R^3's of the surface cut out of the surface a system of nodal quartics. The line p meets a quadric of this sort in the two points of the two R^3's on it, and hence these nodal quartics do not meet p. We have said that the quartic section of (24) by a plane on p is identical with a section of a Weddle surface through a node. The configurations inscribed in the section must be inscribed in the quartic, since the nodal quartic space-curves determining them do not meet p. Hence:

*The quartic section of a Weddle surface through a node admits ∞^1 inscribed configurations B all having the same polarity. This polarity is defined by the conic on the six points of tangency of tangents from the node.**

§ 22. *Further Loci Determined by Basic R^3's.*

We have seen that basic R^3's osculate an m-ic surface f in its $m(m+1)(m+2)$ points of intersection with an $(m+1)$-ic and an $(m+2)$-ic

* This is a degenerate case of a more complicated system of configurations; cf. Morley and Conner, "Plane Sections of a Weddle Surface," *Am. Jour.*, July, 1909.

The surface (22) may be regarded as the section by a space of the spread of bisecants to the elliptic quintic in four dimensions. Most of the facts brought out above are immediately evident from this point of view. Surfaces and curves obtainable in this way will be treated by the author in a future paper, of which an abstract will be found in *Bull. Am. Math. Soc.*, March, 1909.

surface, say f' and f'' respectively. The locus of points of osculation of basic R^3's with surfaces of the net

$$f + \lambda\phi + \mu\psi = 0$$

may be obtained by elimination of λ and μ from this equation and

$$f' + \lambda\phi' + \mu\psi' = 0,$$
$$f'' + \lambda\phi'' + \mu\psi'' = 0.$$

Its equation is

$$\begin{vmatrix} f, & \phi, & \psi \\ f', & \phi', & \psi' \\ f'', & \phi'', & \psi'' \end{vmatrix} = 0, \tag{25}$$

a surface of order $3(m+1)$. Hence:

Basic R^3's osculate surfaces of a net of m-ics in points of a surface of order $3(m+1)$.

In particular:

Points of osculation of basic R^3's with planes on a point lie on a sextic surface This surface evidently contains the point singly. This may be seen geometrically or from (25).

A "higher null-system"* in space has three characteristics: α, the number of null-planes on a point; β, the number of null-points on a plane; γ, the number of times both null-point and null-plane are incident with a given line. Basic R^3's determine such a null-system; the null-points of a plane are points of osculation of basic R^3's with the plane; the null-plane of a point is the osculating plane at the point to the basic R^3 on the point. The characteristics of this null-system are:

$$\alpha = 1, \quad \beta = 6, \quad \gamma = 5.$$

The value of γ is evident from the fact that the locus of null-points of a sheaf of planes is a surface of order $\alpha + \gamma$; we have found that this is 6.

A more convenient form than (25) for the locus of points of osculation of basic R^3's with planes of a sheaf may be found. The basic R^3 on x is

$$y_i = \frac{x_i}{1 - x_i t},$$

* Sturm, "Liniengeometrie," Vol. I, p. 78.

whence
$$\frac{d\,y_i}{d\,t}=\frac{x_i^2}{(1-x_i\,t)^2},$$
$$\frac{d^2\,y_i}{d\,t^2}=\frac{2\,x_i^3}{(1-x_i\,t)^3}.$$

The osculating plane at x, $(t=0)$, is:

$$\left.\begin{aligned}\xi_1&=x_2\,x_3\,x_4\,(x_2-x_3)\,(x_3-x_4)\,(x_4-x_2),\\ \xi_2&=-x_3\,x_4\,x_1\,(x_3-x_4)\,(x_4-x_1)\,(x_1-x_3),\\ \xi_3&=x_4\,x_1\,x_2\,(x_4-x_1)\,(x_1-x_2)\,(x_2-x_4),\\ \xi_4&=-x_1\,x_2\,x_3\,(x_1-x_2)\,(x_2-x_3)\,(x_3-x_1).\end{aligned}\right\}\qquad(26)$$

The locus of points of osculation of basic R^3's with planes of a sheaf on a point x is:

$$x_1\,\xi_1+x_2\,\xi_2+x_3\,\xi_3+x_4\,\xi_4=0,\qquad(27)$$

where the ξ's have the values given in (26). (27) has triple points at the base-points and contains the ten base-lines singly. It also contains the fifteen lines like

$$x_1=x_2-x_3=0.$$

These are the fifteen lines of intersection of the ten base-planes besides the ten base-lines. They are parts of degenerate conics in pencils of conics which form parts of degenerate basic R^3's, and it is geometrically evident that they must lie on (27).

Since some plane on a tangent to an R^3 is the osculating plane at the point of tangency, we have:

(27) *contains the septimic curve of contact of tangents from x to the Weddle surface with nodes at the base-points and x.*

Also:

(27) *touches this Weddle surface along this curve of contact.*

For it meets the Weddle surface in a curve of order 24. Out of this curve the ten base-lines factor. The remaining 14-ic curve must be the 7-ic taken twice, since the Weddle surface is the locus of points of intersection with lines on x of their bisecant basic R^3's, and no bisecant to an R^3 except a tangent can carry an osculating plane.

In a higher null-system with characteristics α, β, γ, the locus of null-points on planes of a pencil is a curve of order $\beta+\gamma$ meeting the axis of the pencil γ times. Hence:

Basic R^3's osculate planes of a pencil in points of a curve of order 11 *meeting the axis, π, of the pencil five times.*

This 11-ic is the partial intersection of two sextic surfaces (27) determined by two points of π. These surfaces meet besides in the twenty-five lines of intersection of base-planes. The 11-ic lies on the cubic surface (16) determined by π. Any cubic surface determined by a line π on x meets (27) in the 11-ic determined by π and in the 7-ic curve of contact of tangents from x to its Weddle surface.

§ 23. *A Quartic Complex Determined by a Quadric.*

If we require that the points of intersection of its bisecant basic R^3 with a line $p_{ij} = \pi_{\kappa l}$ be apolar to a quadric surface

$$(\gamma x)^2 = 0,$$

the line will lie in a complex. Represent as before the pencil of planes on the line by $\alpha + \lambda\beta$, and the range of points by $a + \mu b$. The points of intersection with p of its bisecant basic R^3 are given by the quadratic in μ

$$\Sigma\, a_1 a_2 p_{12} \pi_{12} + \Sigma\,(a_1 b_2 + a_2 b_1)\, p_{12} \pi_{12} \mu + \Sigma\, b_1 b_2 p_{12} \pi_{12} \mu^2 = 0.$$

The points of intersection with γ are given by

$$(\gamma a)^2 + 2\,(\gamma a)(\gamma b)\,\mu + (\gamma b)^2 \mu^2 = 0.$$

The apolarity-condition of these two quadratics is:

$$(\gamma b)^2 \Sigma\, a_1 a_2 p_{12} \pi_{12} - (\gamma a)(\gamma b)\,\Sigma\,(a_1 b_2 + a_2 b_1)\, p_{12} \pi_{12} + (\gamma a)^2 \Sigma\, b_1 b_2 p_{12} \pi_{12} = 0,$$

or, by writing $a_i b_j - a_j b_i = p_{ij}$,

$$\begin{aligned}\overset{6}{\Sigma}\,[\gamma_{33} p_{13} p_{23} + \gamma_{44} p_{14} p_{24} - p_{12}\,(\gamma_{12} p_{12} + \gamma_{13} p_{13} + \gamma_{14} p_{14} \\ + \gamma_{23} p_{32} + \gamma_{24} p_{42}) + \gamma_{34}\,(p_{13} p_{24} + p_{14} p_{23})]\, p_{12} \pi_{12} = 0. \qquad (28)\end{aligned}$$

Hence:

Lines met by their bisecant basic R^3's in a pair of points apolar to a given quadric, lie in a quartic complex.

A special case of this complex is the complex of lines bisecant to basic R^4's and lying in a given space. We shall recur to this in Part IV of this paper.

IV.

BASIC NORM-CURVES IN FOUR DIMENSIONS.

§ 24. *Introductory.*

The base in a space of four dimensions is a set of six points. Given the set of six points, we have the set of ∞^3 rational quartics on them. We shall call quadric spreads of the linear 8-fold system on the six points *basic* quadric spreads. The theorems a), b), c), of Part III, may be modified for the 4-dimensional case as follows:

a′) *Basic quadric spreads meet any space, α, in quadric surfaces apolar to a definite quadric, Q_α, in α.**

b′) *Basic R^4's meet α in 4-points, $\iota_4^{(4)}$, apolar to Q_α.*

c′) *Basic R^4's tangent to α touch at points of Q_α.*

The quadric Q_α is the quadric with regard to which the $(15_6,\ 20_3)$ configuration, Γ say, cut out of the complete 6-point by α, is self-polar. As in Part III we name the points of the base 1, 2, 3, 4, 5, 6: we have then fifteen base-lines, 12, etc.; twenty base-planes, 123, etc.; fifteen base-spaces, 1234, etc. We name the points, planes, and lines of Γ after the lines, spaces, and planes determining them.

As before, an immediate consequence of theorem b′) is:

There is one and only one basic R^4 which meets a given plane, π, three times.

This R^4 is the locus of the pole of π as to the quadrics $Q_{\alpha+\lambda\beta}$ on spaces of the pencil $\alpha+\lambda\beta$ containing π. We shall call it the basic R^4 *trisecant* to π.

We shall make some use of degenerate basic R^4's, and it will be well to point out how they may degenerate. Any base-line 12, taken with an R^3 on 3, 4, 5, 6, and 12/3456, the point where 12 meets 3456, is a degenerate basic R^4. There are also pencils of conics that may be regarded as parts of degenerate basic R^4's. The two planes 123 and 456 have a point 123/456 in common. Any conic on 1, 2, 3, 123/456, taken with a conic on 4, 5, 6, 123/456, is a degenerate basic R^4.

We shall say that an R^3 is *orthic* to a quadric Q when every quadric on the R^3 is apolar to Q, or, which amounts to the same thing, when there is *one* 4-point, and hence ∞^1, inscribed in the R^3 and apolar to Q.† Given an R^3

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* Loria, *loc. cit.* † Reye: "Geometrie der Lage," Part II, p. 226.

orthic to Q, one of these 4-points may be constructed by choosing any point, a, of the R^3 and taking with a the three points of intersection with the R^3 of the polar plane of a as to Q. It is three conditions on an R^3 to be orthic to a quadric, since it is three conditions on the 3-point in the polar plane of a to be apolar to the section of this plane with Q.

§ 25. *The Point-Sets* $\iota_4^{(4)}$.

Project the system of basic R^4's from a base-point, say 6. This gives a system of cubic cones (2-ways). In each of these cones there is a single infinity of basic R^4's, one through every point of the cone. For a basic R^4 cannot meet one of these cones in a point, a, without lying entirely on it, since projection from 1, 2, 3, 4, 5, or a would give an R^3 with seven intersections with a quadric cone. It follows:

There are ∞^2 *such cubic cones projecting basic* R^4'*s from a base-point,* 6. *These cones meet* α *in* R^3'*s of a basic system on* 16, 26, 36, 46, 56.

If every quadric on five points is apolar to a quadric, Q, we may call the set of five points *orthic* to Q. A defining characteristic of such a set of five points is that the polar line as to Q of the line joining any two of the points is on the plane of the opposite three. Now the polar line of 16-26, or 126, as to Q_a, by the well-known properties of Γ, is 345, which is on 36-46-56, or 3456. 16, 26, 36, 46, 56 are, therefore, an orthic 5-point as to Q_α, and every quadric on them is apolar to Q_α. It follows that every R^3 on them is orthic to Q_α. We have from the above the theorem:

4-*points,* $\iota_4^{(4)}$, *in* α *are* 4-*points inscribed in* R^3'*s of the basic system on* 16, 26, 36, 46, 56, *and apolar to* Q_α.

An $\iota_4^{(4)}$ may be constructed when one point, m, is given by drawing the R^3 on 16, 26, 36, 46, 56, m, and finding its three intersections with the polar plane of m as to Q_α. Theorem c′) follows. Also:

Basic R^4'*s osculate* α *in points of a sextic curve* F, *the curve of contact of* R^3'*s on* 16, 26, 36, 46, 56 *with* Q_α.

Basic R^4'*s hyperosculate* α (*have* 4-*point contact*) *in points of osculation of these basic* R^3'*s with* Q_α.

We shall call this set of twenty-four points the points H.

R^3'*s on the six orthic bases of* Γ *define the same curve* F, *and the same twenty-four points of osculation.*

For there are six orthic 5-points in Γ, which we may call the sets 1, ..., 6. Sets $\iota_4^{(4)}$ may be considered as defined by any one of the six systems.

If in S_4 we take as vertices of the pentahedron of reference the five points 1, ..., 5, with 6 as the unit point, we have from Part I:

Basic R^4's touch an m-ic spread in points of its $m(m+1)$-ic 2-way intersection with a spread of order $m+1$.

Basic R^4's osculate an m-ic spread in points of a curve of order $m(m+1)(m+2)$, its curve of intersection with an $(m+1)$-ic and an $(m+2)$-ic spread.

Basic R^4's hyperosculate an m-ic spread in its $m(m+1)(m+2)(m+3)$ points of intersection with an $(m+1)$-ic, an $(m+2)$-ic and an $(m+3)$-ic spread.

In particular:

Basic R^4's touch a space, α, in points of the quadric cut out by

$$(\alpha x^2) = 0, \tag{1}$$

this being an analytical expression for the quadric Q_α.

Basic R^4's osculate α in the points of intersection of Q_α with

$$(\alpha x^3) = 0.$$

*This is a curve, F, of order 6, — the complete intersection of a quadric and a cubic surface in α.**

Basic R^4's hyperosculate α in the twenty-four points, H, cut out of F by

$$(\alpha x^4) = 0.$$

The points H are the twenty-four points of intersection of a quadric, a cubic and a quartic surface in α.

§ 26. *Further Loci in α.*

Several loci in α besides those we have mentioned will be of use to us.

A. We may ask for the locus of the remaining point of intersection with α of basic R^4's osculating α.

B. We may ask for the locus of the points in which basic R^4's bitangent to α touch α. This locus will lie on Q_α.

C. We may ask for the locus of the two further points of intersection with α of basic R^4's tangent to α. C will be a surface; A and B will be twisted curves in α.

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* The curve F is special, but of genus 4. For an account of the properties of such a curve see Pascal, "Rep. der höheren Math.," II, p. 276.

The order of A is easily determined by finding its intersections with a plane of the configuration Γ. A basic R^4 cannot meet a base-space, and hence cannot meet a plane of Γ without degenerating. The R^4's degenerating into conics cannot be made to osculate α. Consider the R^4's made up of the base-line 12 and R^3's on 3, 4, 5, 6, 12/3456. There being six of these R^3's that osculate the plane 3456, the point 12 counts as a 6-fold point of A. Six points 12 are on every plane of Γ. Hence:

The curve A is of order 36.

A is of genus 4, since it is in one-one correspondence with F. It cannot meet Q_α except at points where an R^4 hyperosculates, and hence it must osculate Q_α at each of the twenty-four points H.

The order of B may be determined in a similar way. Only those basic R^4's which degenerate into pairs of conics, say on 123 and 456, can be made to touch α twice and meet a plane of Γ. Each point of contact counts in two R^4's, and hence on each line of Γ there are two nodes of B. There being four lines on each plane of Γ, we have:

The curve B is of order 16.

We now show that B *actually breaks down into two octavics.* Considered in α, B is the locus of points of intersection with lines of Q_α of their bisecant basic R^3's. The locus of points of intersection with lines of a regulus of their bisecant basic R^3's is an octavic on the regulus. For, calling the base in α 1, 2, 3, 4, 5, consider where this curve can meet the plane 123. 45 meets the regulus in two points, thus determining two lines on it. To each of the two points corresponds a point on its line in 123. There are besides, by the same argument, two points of the curve on each of 12, 23, 31. Hence the curve meets 123 in eight points, and is an octavic. It meets a tangent plane to the quadric in eight points: two on one generator and six on the other. It projects from a point of the quadric into a plane octavic curve with one 6-fold point and one double point. Hence it has sixteen apparent double points, and its genus is 5.

B is therefore two octavics, the two curves determined by the two sets of generators on Q_α. Consider the two octavics as projected from a point of Q_α. The thirty-two apparent double points of the two curves, together with their sixty-four intersections, real or apparent, make up the necessary number of double points. There are 2.6.2 apparent intersections along the two generators through the point of Q_α. The other forty intersections are real, two on each line of Γ.

B can meet (αx^3) only where it meets the sextic F, and this can be only in the twenty-four points H. Hence B touches (αx^3) and therefore F at these points.

Each of the two octavics of B is in one-one correspondence with itself, the points of contact of one basic R^4 being considered as corresponding points. Suppose the R^4

$$x_i = \frac{\varkappa_i}{1 - \varkappa_i t}$$

touches α at $t = t_1$ and $t = t_2$. We must then have the following relations:

$$\Sigma \frac{\alpha_i \varkappa_i}{1 - \varkappa_i t_1} = \Sigma \frac{\alpha_i \varkappa_i}{1 - \varkappa_i t_2} = \Sigma \frac{\alpha_i \varkappa_i^2}{(1 - \varkappa_i t_1)^2} = \Sigma \frac{\alpha_i \varkappa_i^2}{(1 - \varkappa_i t_2)^2} = 0. \qquad (2)$$

The coördinates of any point on the line joining the two points of contact may be written:

$$y_i = \frac{\varkappa_i}{1 - \varkappa_i t_1} + \lambda \frac{\varkappa_i}{1 - \varkappa_i t_2}.$$

Now

$$\begin{aligned}(\alpha y^2) &= \Sigma \alpha_i \left(\frac{\varkappa_i}{1 - \varkappa_i t_1} + \lambda \frac{\varkappa_i}{1 - \varkappa_i t_2}\right)^2 \\ &= \Sigma \frac{\alpha_i \varkappa_i^2}{(1 - \varkappa_i t_1)^2} + \frac{2\lambda}{t_1 - t_2} \Sigma \left(\frac{\alpha_i \varkappa_i}{1 - \varkappa_i t_1} - \frac{\alpha_i \varkappa_i}{1 - \varkappa_i t_2}\right) + \lambda^2 \Sigma \frac{\alpha_i \varkappa_i^2}{(1 - \varkappa_i t_2)^2}.\end{aligned}$$

This vanishes for all values of λ on account of the conditions (2). Hence:

The lines joining corresponding points of the curve B lie entirely on the quadric Q_α, which is, as we have seen, obvious from geometrical considerations.

Corresponding points of B coincide at points where basic R^4's hyperosculate. Since B touches F at these points, we have:

*The points H are points of F such that the tangents to F at these points are generators of Q_α; that is, the twenty-four points of tangency of tangents to F that meet F again.**

We find the order of the surface C in a similar way, namely, by finding the order of the curve in which it meets a plane of Γ, say 1234. The R^3's on five points in the space 1234 touch the plane 1234 along a conic, and cut again along a sextic curve which is on C. Further, the four lines 123, etc., count doubly, since any point on the line 123 is on a conic on 1, 2, 3, 123/456, which, taken with either of two conics on 4, 5, 6, 123/456, satisfy our conditions. Hence:

The surface C is of order 14.

It touches Q_α along the sextic F and cuts out the curves B.

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* Pascal, *loc. cit.*

§ 27. *Analytical Treatment of A, B, and C.*

The 14-ic C is analogous to the sextic of theorem e), Part III. We may find its equation referred to an orthic base of Γ by a method similar to that used there in finding the equation of the sextic. Consider 16, 26, 36, 46, 56 as base in α. Take as reference points 16, 26, 36, 46, with 56 as unit point. A basic quadric may be written

$$\Sigma a_{ij} x_i x_j = 0, \tag{3}$$

with the condition

$$\Sigma a_{ij} = 0. \tag{4}$$

The quadric Q_a must be of the form

$$\Sigma a_i \xi_i^2 + 2\lambda \Sigma \xi_i \xi_j = 0, \tag{5}$$

the apolarity-condition of (3) and (5) being merely (4) multiplied by λ. (5) in points is

$$\begin{aligned} Q \equiv \Sigma \left[2\lambda^3 - \lambda^2 (a_2 + a_3 + a_4) + a_2 a_3 a_4 \right] x_1^2 & \\ - 2 \Sigma \lambda (\lambda - a_3)(\lambda - a_4) x_1 x_2 = 0. & \end{aligned} \tag{6}$$

If we find the condition that the polar plane of a point x as to Q touch the basic R^3 on x, we will evidently have the equation of C. The R^3 on x is, parametrically,

$$y_i = \frac{x_i}{1 - x_i t}.$$

This meets the polar plane of x as to Q in the points given by the roots of the cubic in t

$$\left. \begin{aligned} x_1 x_2 x_3 x_4 \Sigma \frac{\partial Q}{\partial x_1} t^3 - \Sigma x_1 x_2 x_3 \left(\frac{\partial Q}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial Q}{\partial x_3} \right) t^2 & \\ + \Sigma x_1 x_2 \left(\frac{\partial Q}{\partial x_1} + \frac{\partial Q}{\partial x_2} \right) t - 2Q = 0. & \end{aligned} \right\} \tag{7}$$

(7) may be written

$$x_1 x_2 x_3 x_4 L t^3 - f_4 t^2 + f_3 t - 2Q = 0, \tag{8}$$

where L is the polar plane of 56 as to Q, and f_3 and f_4 are a cubic and a quartic surface respectively. (8) gives the parameters along the R^3 on x of the other three points of the set $\iota_4^{(4)}$ determined by x. The parameter of x is 0. Hence we have:

Basic R^4's touch α where $Q = 0$.

Basic R^4's osculate α where $Q = f_3 = 0$.

Basic R^4's hyperosculate α where $Q = f_3 = f_4 = 0$.

In other words, f_3 is a cubic surface on the curve F, and f_4 is a quartic surface cutting out of F the points H.

C is the discriminant of (8):

$$C \equiv 108\, x_1^2 x_2^2 x_3^2 x_4^2 L^2 Q^2 + 4\, x_1 x_2 x_3 x_4 L f_3^2 - 36\, x_1 x_2 x_3 x_4 L Q f_3 f_4 + 8 f_4^3 Q - f_3^2 f_4^2 = 0. \qquad (9)$$

C is of the form

$$\Phi Q + \psi f_3^2 = 0,$$

and hence touches Q along its intersection with f_3. The further intersection with Q is the curve B. B is therefore cut out of Q by

$$\Psi = 4\, x_1 x_2 x_3 x_4 L f_3 - f_4^2 = 0.$$

This form shows that B touches F at the points H; in fact Ψ touches f_3 all along the curve cut out by f_4. The form of Ψ shows that B has nodes at its intersections with the planes of Γ.

The curve A must lie on C. We can find from (8) a surface that will cut it out. If x is a point of A, (8) must have three equal roots, and its Hessian must vanish identically. This Hessian is

$$\phi_8 t^2 + \phi_7 t + \phi_6 = 0,$$

where

$$\begin{aligned} \phi_8 &= 3\, x_1 x_2 x_3 x_4 L f_3 - f_4^2, \\ \phi_7 &= 18\, x_1 x_2 x_3 x_4 L Q - f_3 f_4, \\ \phi_6 &= 6 f_4 Q - f_3^2. \end{aligned}$$

ϕ_6 is the locus of point-sets $\iota_4^{(4)}$ which are a self-apolar quartic on the basic R^4 (or basic R^3) determining them.* The g_3 of this quartic is

$$9 f_3 f_4 Q - 54\, x_1 x_2 x_3 x_4 L Q^2 - f_3^3 = 0.$$

This is the locus of sets $\iota_4^{(4)}$ that are harmonic pairs on the R^4 determining them. It osculates Q along the sextic F, as is evident geometrically. The above equation has the form

$$f_3 \phi_6 - 3\, Q \phi_7 = 0.$$

Hence g_3 contains the curves common to ϕ_6 and ϕ_7.

A is the curve common to ϕ_6, ϕ_7, and ϕ_8. ϕ_6 and ϕ_7 meet in A and the curve F. ϕ_6 has nodes at the points of the orthic 5-plane $x_1 x_2 x_3 x_4 x_5 L$, and ϕ_7 has triple points at these points. A, therefore, has 6-fold points at the points

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* See Part V, § 36.

of Γ. ϕ_6 and ϕ_7 meet in A and the curve common to f_3 and f_4. C is obviously, to within a factor,

$$4\,\phi_6\,\phi_8 - \phi_7^2 = 0.$$

ϕ_6, then, touches C along the curves F and A. Indeed, the form of ϕ_6 shows that it touches Q, and hence C, along the curve F.

C may also be put into the form

$$x_1\,x_2\,x_3\,x_4\,L\,\Phi + \Psi\,f_4^2 = 0.$$

This shows that f_4 meets C at its points of intersection with $x_1\,x_2\,x_3\,x_4\,L$. These five planes form an orthic 5-plane of Γ. It may be easily verified that f_4 has nodes at the ten points of this 5-plane. Hence:

The points H are on a 10-nodal quartic surface containing the ten lines of any orthic 5-plane of Γ and with nodes at the points of this 5-plane. There are six such quartic surfaces.

The surface f_3 may easily be shown to be on the ten points of an orthic 5-plane. Hence:

The curve F is on a cubic surface with the ten points of any orthic 5-plane of Γ.

§ 28. *A Cubic Spread Determined by a Plane in S_4.*

We see from (1) that quadrics $Q_{\alpha+\lambda\beta}$ on spaces of the pencil,

$$(\alpha\,x) + \lambda\,(\beta\,x) = 0,$$

are cut out by the quadric spreads

$$(\alpha\,x^2) + \lambda\,(\beta\,x^2) = 0.$$

Eliminating λ between these two equations we have, as the locus of quadrics $Q_{\alpha+\lambda\beta}$,

$$\begin{vmatrix} (\alpha\,x^2), & (\beta\,x^2) \\ (\alpha\,x), & (\beta\,x) \end{vmatrix} \equiv \Sigma\,\pi_{ij}\,x_i\,x_j\,(x_i - x_j) = 0,\text{*} \qquad (10)$$

where $\pi_{ij} = \alpha_i\,\beta_j - \alpha_j\,\beta_i$ are the coördinates of the plane common to the spaces of the pencil. It is easy to show that the basic R^4 on x passes through x with the tangent

$$p_{ij} = x_i\,x_j\,(x_i - x_j).$$

Hence we have:

The locus of points of tangency with basic R^4's of tangent lines meeting a plane π is the cubic spread (10).

---

* This seems to be the *general* cubic spread in S_4 which contains a plane.

In what follows we shall denote this spread by $[\pi]$. It contains the plane π singly.

Consider two cubic spreads $[\pi]$ and $[\pi']$, where the planes π and π' have a line, p, in common. They intersect in the quadric Q_a on the space α containing π and π', and in a septimic 2-way $[p]$, the locus of points of tangency with basic R^4's of tangents meeting the line p. Hence:

The locus of points of tangency with basic R^4's of tangent lines that meet a given line, p, is a 7-ic 2-way, $[p]$.

$[p]$ is met by a space α on p in p and a sextic curve on Q_a. This sextic curve is easily defined by the basic R^3's in α. It is the intersection with Q_a of the cubic surface (16), Part III, determined by the line p. It follows that the sextic curve meets p twice.

Consider now a line, p, and a plane, π, meeting it in a point, a. The spreads $[p]$ and $[\pi]$ meet in a 21-ic curve. Out of this curve the above sextic on the space πp factors. The residual 15-ic curve is the locus of points of tangency of basic R^4's with tangents passing through a. Hence:

The locus of points of tangency with basic R^4's of tangents passing through a point, a, is a 15-ic curve $[a]$.

$[\pi]$ contains π, $[p]$ contains p, and $[a]$ contains a, each singly. Each locus is basic.

§ 29. *The Congruence of Tangents to Basic R^4's.*

There are ∞^1 lines tangent to a basic R^4, and ∞^3 basic R_4's; ∞^4 lines in all. Hence these lines are in a congruence. The order of $[a]$ shows that the order of this congruence is 14, since the congruence cone on any point a is the cone projecting $[a]$ from a. Congruence lines in a space α are tangents to basic R^3's in α at points of a quadric Q_a. Congruence lines meeting a line p are tangents to basic R^3's at points of a sextic curve on Q_a. By a theorem of Part III, these lines lie on a ruled surface of order $6.3=18$. Now the lines of an (m, n) congruence that meet a given line, p, lie on a ruled surface of order $m+n$. The class of our congruence is therefore 4. We have:

There are four basic R^4's that touch a given plane, π.

Lines tangent to basic R^4's and in a space α are in a $(14, 4)$ congruence.

We next determine its *rank*. It is known that the lines of an (m, n, r) congruence that meet a line p lie on a ruled surface of order $m+n$, with p as an m-fold line, and that this surface has an additional double curve of order

$\frac{n(n-1)}{2}+r$. Cut the 18-ic surface, on a space α and determined by the lines of our congruence meeting a line, p, by a plane π on α. This gives an 18-ic curve on π which is in one-one correspondence with the sextic curve of contact of basic R^4's with the lines of our surface, and hence is of genus 4. This 18-ic has a 14-fold point and besides

$$\tfrac{1}{2} . 17 . 16 - \tfrac{1}{2} . 14 . 13 - 4 = 41$$

nodes. These are the points in which π meets the additional double curve of the surface, this curve being of order $6+r$. Hence:

The rank of the congruence of lines tangent to basic R^4's and on any space α, is 35.

From what we have said we have the further theorem:

The locus of tangents at points of a line p to basic R^4's meeting p is a cubic 2-way having p for a directrix.

For the line p meets a cubic spread $[\pi]$ in three points, and these points determine the points in which π meets our 2-way. Also:

The locus of tangents at points of a plane π to basic R^4's meeting π is a ruled spread of order 7.

For π meets a 7-ic 2-way $[p]$ in seven points, and the tangents to basic R^4's at these points determine the points in which p meets our spread.

§ 30. *Loci Determined by Systems of Spreads.*

The points of contact of basic R^4's with spreads of the pencil of m-ics,

$$f+\lambda\phi=0,$$

are cut out by $(m+1)$-ic spreads

$$f'+\lambda\phi'=0.$$

Eliminating λ, we have the $(2m+1)$-ic spread

$$\begin{vmatrix} f, & \phi \\ f', & \phi' \end{vmatrix}=0.$$

Hence:

Points of contact of basic R^4's with spreads of a pencil of m-ics are on a spread of order $2m+1$.

Similarly:

Points of osculation of basic R^4's with spreads of a net of m-ics,

$$f+\lambda\phi+\mu\psi=0,$$

are on a spread of order $3m+3$, *of the form*

$$\begin{vmatrix} f, & \phi, & \psi \\ f', & \phi', & \psi' \\ f'', & \phi'', & \psi'' \end{vmatrix} = 0. \tag{11}$$

Let us apply this to a particular case. If f is a basic quadric spread, the curve of osculation of basic R^4's with f, Φ say, is a curve of order 24. But a basic R^4, having already six points in common with f, cannot osculate without lying entirely on f. Φ must therefore be six basic R^4's. We have, then:

There are six basic R^4*'s on a basic quadric spread.*

We should expect, by counting of constants, to find a finite number of basic R^4's on a basic quadric spread. For it is nine conditions on an R^4 to lie on a quadric spread, and the spread therefore contains $\infty^{21-9} = \infty^{12}$ R^4's. These twelve remaining degrees of freedom are taken up by requiring the R^4's to be on six specified points of the spread, since it is two conditions on a curve in a 3-dimensional space (linear or not) to be on a point. This gives us at once:

If f, ϕ, *and* ψ *are basic quadric spreads,* (11) *is the locus of basic* R^4*'s lying on spreads of the net* $f+\lambda\phi+\mu\psi=0$. (11) *is in this case of order* 9.

Let us further specialize by making $f+\lambda\phi+\mu\psi=0$ the net of quadric spreads on a plane π and the six base-points. All spreads of the net must contain the basic R^4 trisecant to π, since this R^4 meets any of these spreads in $6+3=9$ points. Now a quadric spread in S_4 containing a plane must have a node and contain two systems of planes on the node, and an R^4 on it meets all planes of one system three times and of the other system once. All R^4's on the spread (11), then, meet π. Again, a basic R^4 meeting π must be contained in some quadric spread of the net, namely that particular quadric spread containing any two further points of it. Hence:

The locus of basic R^4*'s meeting a plane* π *is a spread of order* 9. *It has the trisecant basic* R^4 *to* π *for a triple curve. Its equation is of the form* (11).

It follows that:

There are nine basic R^4*'s meeting a plane and a line.*

Basic R^4*'s meeting a line lie on a 2-way of order* 9.

§ 31. *Some Enumerative Results.*

There being ∞^3 basic R^4's, there are in general a finite number satisfying a given 3-fold condition. We can, by means of the knowledge we have of the

curves A and B and the surface C, taken with the theorems just enunciated, find the number of basic R^4's satisfying a variety of 3-fold conditions. We introduce the following symbolism. Denote by:

P_6 the 18-fold condition that an R^4 be basic,
P the 3-fold condition that an R^4 be on a seventh point,
ν the 2-fold condition that an R^4 meet a given line,
μ the 1-fold condition that an R^4 meet a given plane,
ρ the 3-fold condition that an R^4 touch a given plane,
α_2 the 1-fold condition that an R^4 touch a given space,
α_3 the 2-fold condition that an R^4 osculate a given space,
α_4 the 3-fold condition that an R^4 hyperosculate a given space.

A basic R^4 subjected to a λ-fold condition τ has for locus a $(4-\lambda)$-way spread which we shall denote by $[\tau]$. By the theorem of Sturm enunciated in Part III, two spreads $[\tau]$ and $[\tau']$ can intersect only in basic R^4's and in parts of basic R^4's — base-planes, lines, or conics. In order to find the order of the spread of intersection of two of these spreads apart from base-lines, conics on base-planes, and base-planes, it is necessary to know the multiplicity of the base-lines, conics and planes in these spreads. The multiplicity of the base-points in a given 3-way is immediate from the fact that basic R^4's cannot meet the spread except at base-points without lying entirely on it. The following table gives the multiplicities of base-lines, planes and conics in a number of these spreads. Accented letters are meant to indicate that the symbol repeated refers to a different element; indices, that the same element is repeated.

Spread.	Dimension.	Mul. of 15 base-lines.	Mul. of 20 base-planes.	Base-Conics.	Order.
$[\nu]$	2	1	0	0	9
$[\mu]$	3	3	1	0	9
$[\alpha_2]$	3	6	2	0	18
$[\alpha_3]$	2	6	0	0	54
$[\alpha_2^2]$	2	0	0	2 double conics on each plane	32
$[\mu^2]$	2	1	0	1 conic on each plane	17
$[\mu\mu']$	2	5	0	2 conics on each plane	61
$[\alpha_2\mu]$	2	10	0	1 double conic and 2 single conics on each plane	122
$[\alpha_2\alpha_2']$	2	20	0	4 double conics on each plane	244

We now verify the facts contained in this table.

We have shown the order and dimension of $[\nu]$ to be 9 and 2 respectively. It contains the base-lines singly, since the unique R^3 on say 1, 2, 3, 4, 56/1234, and the point where the line ν meets 1234, taken with 56, lies in the spread.

We have seen also that $[\mu]$ is a 3-way of order 9. The plane μ meets a base-plane 123 a point, and the conic of the pencil on 123 on this point, taken with any conic of the pencil on 456, is a basic R^4 meeting μ. A pencil of conics covers a plane singly (there is a unique conic on every point of the plane) and hence $[\mu]$ contains the base-lines singly. $[\mu]$ and $[\nu]$ must meet in nine basic R^4's, the nine R^4's meeting μ and ν. They meet in a curve of order 81. This leaves a curve of order $81 - 36 = 45$ to be accounted for by the fifteen base-lines. Hence $[\mu]$ must contain the base-lines triply, since $[\nu]$ contains them singly.

$[\alpha_2]$ is a 3-way of order 18, since $[\nu]$ meets Q_α in eighteen points and the basic R^4's on these points determine eighteen points in which ν meets $[\alpha_2]$. Two conics of the pencil on 123 touch α, and hence every conic of the pencil on 456 counts doubly in this spread. $[\alpha_2]$ and $[\nu]$ meet in eighteen basic R^4's. They meet in a curve of order 162, in which the base-lines form a part of order $162 - 72 = 90$. Hence $[\alpha_2]$ contains the base-lines 6-fold. It follows:

The points of the configuration Γ in α are sextuple points of the surface C.

$[\alpha_3]$ is a 2-way of order 54, since $[\mu]$ meets the sextic F in α in fifty-four points. Six R^3's of the basic system in 1234 osculate α. Hence the base-lines are 6-fold.

$[\alpha_2^2]$ meets α in B taken twice. Its order is therefore 32. Two conics of each of the pencils 123 and 456 touch α and these must count doubly in $[\alpha_2^2]$.

$[\mu^2]$ contains the base-lines singly, since there is a unique basic R^3 in 1234 bisecant to the line in which μ meets 1234. It contains on each plane the conic on the point in which μ meets the plane. If x is its order, we have, by combination with $[\mu]$,

$$9x - 15 \,.\, 3 - 20 \,.\, 2 = 4x, \text{ whence } x = 17.$$

$[\mu\mu']$ contains the base-lines 5-fold, since five basic R^3's meet two lines. It contains the two conics on the points in which μ and μ' meet a base-plane. If x is its order, we have, by combination with $[\mu]$,

$$9x - 15 \,.\, 3 \,.\, 5 - 4 \,.\, 20 = 4x, \text{ whence } x = 61.$$

$[\alpha_2 \mu]$ contains the base-lines 10-fold, since ten basic R^3's meet a line and touch a plane.* Two conics on 123 and two on 456 touch α. Either of the tangent conics on 123 taken with the conic on 456 on the point in which μ meets 456 is a basic R^4 satisfying the conditions. The latter counts doubly, the former singly. Combination with $[\mu]$ gives the order.

$[\alpha_2 \alpha_2']$ contains the base-lines 20-fold, since twenty basic R^3's touch two planes.* Two conics on each base-plane touch each space α and α', and hence there are four double conics on each base-plane. Combination with $[\mu]$ gives the order.

The following table gives the number of basic R^4's satisfying various 3-fold conditions. Most of these may be obtained by combination of the appropriate spreads given in the foregoing table, together with the application of the theorem of Sturm. Others are added for the sake of completeness.

$$\begin{array}{llll}
P_6 P = 1, & P_6 \alpha_2 \nu = 18, & P_6 \alpha_2^2 \mu = 32, & \\
P_6 \mu^3 = 1, & P_6 \alpha_3 \mu = 54, & P_6 \alpha_2^2 \alpha_2' = 64, & \\
P_6 \rho = 4, & P_6 \alpha_2 \alpha_3' = 108, & P_6 \mu \mu' \mu'' = 61, & P_6 \alpha_2 \alpha_2' \alpha_2'' = 488. \\
P_6 \alpha_4 = 24, & P_6 \mu^2 \mu' = 17, & P_6 \alpha_2 \mu \mu' = 122, & \\
P_6 \mu \nu = 9, & P_6 \mu^2 \alpha_2 = 34, & P_6 \alpha_2 \alpha_2' \mu = 244, &
\end{array}$$

The fact that some of these results may be obtained in more than one way furnishes an easy check on the accuracy of the first table.

We may add to the above the following extensions of theorems of Part III:

Basic R^4's meeting a 2-way of order m lie on a 3-way of order $9m$.

Basic R^4's meeting a curve of order m lie on a 2-way of order $9m$.

Modifications of these theorems will be necessary if the director spread is related to the base.

§ 32. *Loci Determined on a Plane by Basic R^4's.*

Consider a pencil of spaces

$$(\alpha x) + \lambda (\beta x) = 0,$$

having a plane, S, as axis. The quadric spreads cutting out the quadrics $Q_{\alpha+\lambda\beta}$ on spaces of this pencil are

$$(\alpha x^2) + \lambda (\beta x^2) = 0.$$

It follows that the quadrics $Q_{\alpha+\lambda\beta}$ cut out on S a pencil of conics, Φ say. The

* Sturm, *loc. cit.*

three points of intersection with S of its trisecant basic R^4 are apolar to all quadrics $Q_{\alpha+\lambda\beta}$, by theorem c'). They must therefore be apolar to all conics Φ, and so must be the vertices of the diagonal triangle of the pencil Φ.

If a basic R^4 touches S it touches every space of the pencil on S. Hence:

The four basic R^4's tangent to S touch at the base-points of the pencil Φ.

The base-planes and spaces meet S in a configuration, Δ say, made up of twenty points and fifteen lines, four points on a line, three lines on a point. Δ is, by the Veronese "Perspective Pyramid Theorem," the figure of three triangles in continued perspective, taken with their axes of perspection,—these three axes meet in a point. Since the lines (on a space $\alpha+\lambda\beta$) 123 and 456 are polar lines in the quadric $Q_{\alpha+\lambda\beta}$, it follows that the points 123 and 456 are apolar to every conic Φ. These results easily give the following theorem:

*The six conics defined by the six configurations B of Δ are in a pencil Φ, the base-points of which are the points of tangency of tangent basic R^4's to S. The vertices of the triangle apolar to Φ are the points of intersection with S of its trisecant basic R^4.**

Basic R^4's meeting a line, p, lie on a 2-way, P, of order 9. Any space, α, on p meets P in a nonic curve, the line p and an octavic curve P' in α. Any plane S on p and in α meets P' in eight points. Now P' meets p in two points, the two points on p in which basic R^4's touch α. Hence there are six basic R^4's that meet a line p on S and meet S at some other point not on p. Therefore:

The locus of points of intersection with a plane, S, of its bisecant basic R^4's is a sextic curve, Σ.

Σ has obviously three nodes, at the points of intersection with S of its trisecant basic R^4. It can in general have no further nodes, since this would require two basic R^4's on a point of S. Its genus is therefore 7. It contains the twenty points of the configuration Δ, conics of the pencils on 123 and 456 and on the points where S meets these planes, forming a basic R^4 bisecant to S. It also contains the base-points of the pencil Φ, and touches at these points the tangents to basic R^4's through them.

Consider the set of basic R^4's as projected from a base-line, 12 say. They are projected by quadric spreads with 12 as a nodal line, and project into conics on S on 123, 124, 125, 126. Points of intersection with S of bisecant basic R^4's must be on conics of this pencil. They must further be apolar to all conics Φ. Hence:

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* Carver: *Trans. Am. Math. Soc.*, Vol. VI, No. 4, p. 543.

Points of intersection with a plane, S, of its bisecant basic R^4's are partners in the Cremona involution determined by the pencil Φ associated with the configuration Δ on S. They are also on conics of fifteen pencils on 4-points of Δ like 123, 124, 125, 126.

We saw in Part II that the locus of pairs of points determined in this way is a sextic curve. The involution determined by conics Φ leaves this sextic unaltered. It carries the lines of Δ into conics on L, M, N, the diagonal 3-point of Φ. It carries 123 into 456, etc., as we have seen, thus leaving the points of Δ, as a whole, unaltered. Hence:

The points 123, 124, 125, 126, L, M, N *are on a conic.*

This is the conic into which the basic R^4 trisecant to S is projected from 12. Δ determines in this way a configuration of conics with the same set of points and the same incidence conditions.

We also proved in Part II that the locus of joins of corresponding points on the sextic Σ is a line-quartic, W say. W touches the six lines of both bases Φ and Φ', the second generating pencil Φ' being conics on 123, 124, 125, 126. Since W is symmetrically related to Δ, it touches all the twenty lines of Δ. Since 123 and 456 are corresponding points of Σ, W touches the ten lines 123/456. It obviously touches the three lines of the diagonal triangle of Φ. But it is more specially related to these lines, as we shall see.

The configuration Δ_1 dual to Δ determines a range of conics Φ_1. Here we have a line-sextic Σ_1 and a point-quartic W_1. The configuration Δ_1 may be considered as the section by a plane of the complete 6-point in three dimensions. Here W_1 is the section by S of the Weddle surface with nodes at the six points. The diagonal 3-point of the range Φ_1 is the three points in which the R^3 on the six points in space meet the plane. The properties of the plane section of a Weddle quartic* may be here stated for the sextic Σ and the quartic W:

The line-quartic W determined by basic R^4's on a plane S touches the three lines of the diagonal triangle of the pencil Φ in three points on a line. The two tangents to W from a point of the diagonal 3-point, besides the two diagonal lines on the point, are apolar to these two lines. The configuration Δ is not unique on W, there being an infinity of such configurations whose lines touch W and whose points are on Σ.

The first part of this theorem may be verified easily from the form of the equation given in Part II. The specialization here is that one of the conics

$$(\alpha x)^2 + \lambda(\beta x)^2 = 0$$

* Morley and Conner, *loc. cit.*

is on the reference 3-point. Let this be $(\alpha x)^2$. We have, then,

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0,$$

and equation (13), Part II, becomes

$$\Sigma\, \alpha_{12}\, \beta_{33}\, \xi_1\, \xi_2\, (\xi_1^2 - \xi_2^2) - 2\, \Sigma\, (\alpha_{12}\, \beta_{13} - \alpha_{13}\, \beta_{12})\, \xi_1^2\, \xi_2\, \xi_3 = 0,$$

from which the first part of the above theorem is an easy inference.

Let us return for a moment to the system of quadrics on six points of space. It is three linear conditions on a quadric to touch a given plane at a given point. Hence one quadric on the six points touches a plane, S, at a given point. The two lines cut out of this quadric by S are partners in the Cremona line-involution determined by the range Φ_1 on S. At a point, a, of the quartic W_1, the quadric tangent to S is a cone with vertex at a. The two lines cut out of this cone by S are lines of the sextic Σ_1. One quadric on the six points contains an arbitrarily chosen line of space. If we require this quadric to have a node, the line lies in a complex. We have, therefore:

Generators of quadric cones on six points of space are in a sextic complex. The complex curve on any plane, S, is the sextic Σ_1. This complex contains the (1, 3) *congruence of bisecants to the R^3 on the six points doubly.**

The range Φ_1 of the configuration Δ_1 determines a conjugate 3-fold system of conics to which all conics Φ_1 are apolar. If the plane α is mapped by this scheme on a Steiner quartic surface, Ω, the lines 123, 456, being conjugate lines in the range, will go by this mapping scheme into a tangent plane section of Ω, touching at 123/456. We get in this way ten planes and fifteen points. If we indicate the points of Δ_1 by two-figure symbols, the incidence relations of these ten planes and fifteen points may be exhibited in the two diagrams:

x	46	62	24		135	x	x	x
35	12	14	16		x	146	162	124
51	32	34	36		x	346	362	324
13	52	54	56		x	546	562	524.

135 in the second diagram stands for 135/246, etc. A point in the left-hand diagram is incident with the planes in the right-hand diagram standing in its row and column, except the one in its position, and similarly for the planes. But this is exactly the scheme of nodes and tropes of a 15-nodal quartic surface. Therefore:

* Sturm: "Die Lehre von den Geom. Verwandtschaften," III, p. 409.

Conics apolar to the range Φ_1 of Δ_1 map the plane into a Steiner quartic surface Ω. The points of Δ_1 go into fifteen points of Ω. The lines 123, 456, *etc., go into ten tangent plane sections of Ω. The points and planes thus obtained form the configuration of nodes and tropes of a* 15*-nodal quartic surface, fifteen points lying by sixes on ten conics.*

§ 33. *Complex of Lines Bisecant to Basic R^4's.*

We proved at the beginning of Part IV that two points in which a basic R^4 meets a space α are points of an R^3 of a basic system in α and apolar to Q_α. At the end of Part III we showed that the lines joining pairs of points thus determined are in a quartic complex. Hence:

Lines bisecant to basic R^4's are in a quartic complex.

It is obvious that these lines may also be defined as the lines met by quadrics $Q_{\alpha+\lambda\beta+\mu\gamma}$ of the net of spaces on them in pairs of an involution. This property makes the finding of the equation of the four-dimensional complex easy by methods similar to those employed in Part III. The following is also obvious:

The complex curve on any plane, S, is the quartic W.

We see from equation (28), Part III, that if $p_{12}=p_{13}=p_{14}=0$, involving $\pi_{34}=\pi_{42}=\pi_{23}=0$, or $p_{12}=p_{23}=p_{31}=0$, involving $\pi_{34}=\pi_{14}=\pi_{24}=0$, the equation of the complex is satisfied. Hence the complex (28) contains every line on a base-point or on a base-plane. Since our complex is symmetrically related to the configuration Γ in α, we have:

The section by a space α of the complex of lines bisecant to basic R^4's contains singly every line on the planes of the configuration Γ in α, and every line on its points. This amounts to saying: The complex of bisecants to basic R^4's contains, in general singly, every line meeting a base-line, and every line lying on a base-space.

This is also evident directly. A line on a base-space has a unique bisecant basic R^3 of the system in that space, and hence a degenerate bisecant basic R^4. A line meeting 12 meets on R^3 of the basic system on 3456, the R^3 on the point where it meets 3456, and hence has a degenerate bisecant basic R^4. A similar argument shows that our quartic complex contains every line meeting opposite planes 123/456, etc.

It was shown by Sturm that the complex of tangents to basic R^3's contains lines on base-points and base-planes doubly. The section of our complex with the sextic complex determined by 16, 26, 36, 46, 56, as base, leaving out of

account the above-mentioned degenerate congruences, is a congruence of order $6.4-5.2=14$, and of class $6.4-10.2=4$. This is the (14, 4) congruence of tangents to basic R^4's.

§ 34. *An 11-ic 2-way Determined by Seven Points.*

Consider any plane, S, and a space, α, on it. Basic R^4's bisecant to S meet α in two other points, and the lines joining these pairs of points form the complex cone on α of bisecants to basic R^4's through M, the pole of S as to Q_α. The locus of these two extra points of intersection is a curve in S with a triple point at M, since the trisecant basic R^4 to S counts as three bisecant R^4's, and this R^4 passes through M by theorem c′). This locus is met by a plane in α through M in eight points on the cone proper, and in three points at M. It is obviously the locus of points in α from which 1, 2, 3, 4, 5, 6, M are projected into seven points of an R^3, since an R^4 projects into an R^3 from any point on it. Hence:

The locus of points in S_4 from which seven points project into seven points of an R^3 is a 2-way, Ψ of order 11, with triple points at each of the seven points.

The surface Ψ shows a close analogy with the Weddle surface in S_3. Let us project it from one of its triple points, M say, into a space β. Consider any point, $\varkappa$, of the surface. If we project 1, 2, 3, 4, 5, 6 from $\varkappa$ on a space γ through M, the points 1γ, 2γ, 3γ, 4γ, 5γ, 6γ * are on an R^3 with M in γ, and are projected from M by six lines of a quadric cone, V, in γ. Now the lines $\varkappa 1$, $\varkappa 2$, etc., project from M into β into the lines $\varkappa_\beta 1_\beta$, $\varkappa_\beta 2_\beta$, etc. Since $\varkappa 1$, $\varkappa 2$, etc., meet γ in points of the R^3 $1\gamma, \ldots, M$, and hence in points of V, $\varkappa_\beta 1_\beta$ must meet β_γ in points of the conic V_β. In other words, 1_β, 2_β, etc., project from $\varkappa_\beta$ on the plane β_γ into six points of a conic. Hence:

The surface Ψ projects from any one of its triple points into a Weddle surface taken twice.

The 11-ic section of Ψ on our space through M projects into a plane section of this Weddle surface. We have, then:

The section by a plane in α of the cone of the complex of bisecants to basic R^4's may be regarded as a plane section of a Weddle surface, a theorem which we have met in another connection.

The 11-ic curve which is the section of the surface Ψ by a space α on M, is the section (in α) of the Weddle surface with nodes at 16, 26, 36, 46, 56, M,

* 1γ is meant to indicate the point into which 1 projects.

with the complex cone of bisecants to basic R^4's. The complementary intersection is the five lines $M16$, $M26$, $M36$, $M46$, $M56$. There are six such Weddle surfaces corresponding to the six orthic bases of Γ, all containing the curve. Two, M, 16, 26, 36, 46, 56 and M, 15, 25, 35, 45, 65, intersect in this curve and 156, 256, 356, 456, $M56$.

We have said that Ψ has triple points at each of its seven defining points, 1, 2, 3, 4, 5, 6, 7, say. It contains the R^4 on the seven points, since this R^4 projects into an R^3 from any one of its points. It touches the osculating planes of this R^4, since the Weddle surface into which it projects from any one of its seven triple points touches the osculating planes of the R^3 on its six nodes,—the curve into which the R^4 projects. Ψ contains the twenty-one lines 12, etc., since projected from a point of one of these lines the seven points become six, and an R^3 can be put on six points in an S_3. Ψ contains the thirty-five lines where 123 meets 4567, etc., since 1, 2, 3 project from one of these lines into three points on a line, and the conic on the four points into which 3, 4, 5, 6 project and the point in which the line meets their plane, taken with the line, is a degenerate R^3.

V.

BASIC NORM-CURVES IN n DIMENSIONS.

§ 35. *General Notions.*

The theorems which we shall give in this part of our paper are mainly of an enumerative character, but their value in the treatment of higher cases is sufficiently obvious.

We shall indicate by S_m a linear space or flat of m dimensions, and by C_m^p a spread of m dimensions and of order p. If no ambiguity can result, spreads arising in the course of the discussion will frequently be referred to by the symbol C_m^p, indicating their order and dimension.

The base in S_n is a set of $n+2$ points. We have as before the $(n-1)$-fold system of basic R^n's, and the $\frac{1}{2}(n^2+n-4)$-fold system of basic quadric spreads, C_{n-1}^2's. Theorems a), b), c), at the beginning of Part III, may here be stated:

a′′) *Basic C_{n-1}^2's meet any S_{n-1}, α, in C_{n-2}^2's apolar to a definite C_{n-2}^2, Q_α, in α.*

b′′) *Basic R^n's meet α in sets of n points, $\iota_n^{(n)}$, apolar to Q_α.*

c′′) *Basic R^n's tangent to α touch at points of Q_α.*

Q_a is the quadric spread in α with regard to which the configuration Γ^n, cut out of the complete $(n+2)$-point by α, is self-polar. Naming the base-points and the elements of this configuration as before, this means that $345\ldots(n+2)$ is the polar S_{n-2} of 12 as to Q_a, and so for the other points and S_{n-2}'s of Γ^n.

The configuration Γ^n contains $\binom{n+2}{2}$ points, $\binom{n+2}{3}$ lines, ..., $\binom{n+2}{m+2}$ S_m's. Every S_m contains $\binom{m+2}{p+2}$ S_p's, and every S_p is incident with $\binom{n-p}{m-p}$ S_m's, where $p < m$.*

We have from theorem b''), as before:

There is one and only one basic R^n meeting an S_{n-2} $n-1$ times.

We may add to this the following:

The basic R^n meeting an S_{n-2} $n-1$ times is common to the system of basic C^2_{n-1}'s containing the S_{n-2}.

For the R^n meets one of these C^2_{n-1}'s in $n+2+n-1=2n+1$ points, and must therefore lie on it.

We shall call an S_p and an S_q of the complete $(n+2)$-point *opposite* if $p+q=n$, and S_p and S_q have no base-point in common. Opposite spaces have one point in common. Basic R^n's may degenerate into an R^p in S_p on the $p+1$ base-points in S_p and the point in which S_p meets its opposite S_q, taken with a similarly determined R^{q} in S_q. Further degeneration may occur, but this is the simplest, and by far the most important for our purpose. We can have further degeneration only by the breaking up of R^p or R^q in a similar way. From these facts we have the theorem:

Every base S_p carries a basic system of R^p's on the $p+1$ base-points in S_p and the point in which S_p meets its opposite S_q. All these R^p's are to be regarded as parts of degenerate basic R^n's, the complementary R^q's being the system of R^q's on the similarly determined base in the opposite S_q.

§ 36. *The Point-Sets $\iota_n^{(n)}$.*

We shall say that an R^n is *orthic* to a C^2_{n-1}, Q, when every C^2_{n-1} on the R^n is apolar to Q. This is equivalent to saying that there is one, and hence ∞^1, $(n+1)$-points inscribed in the R^n and apolar to Q. One of these $(n+1)$-points may be constructed by choosing a point, a, of R^n, and taking with a the n points

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* Carver, *loc. cit.*

in which the R^n meets the polar S_{n-1} of a as to Q. It is $2n+1$ linear conditions on a C^2_{n-1} to contain a given R^n. There are then

$$\tfrac{1}{2}n(n+3)-2n-1+1=\tfrac{1}{2}n(n-1)$$

linearly independent C^2_{n-1}'s on a given R^n. It is, then, $\frac{1}{2}n(n-1)$ conditions on the system of quadrics to be apolar to Q, and hence it is $\frac{1}{2}n(n-1)$ conditions on the R^n to be orthic to Q.

Basic R^n's are projected from a base-point by a system of ∞^{n-2} C_2^{n-1}'s. There are ∞^1 basic R^n's on each of these cones, one through every point of the cone. These C_2^{n-1}'s meet α in a basic system of R^{n-1}'s on an orthic base of Γ^n. We mean by an "orthic base" of Γ^n a set of $n+1$ points of Γ^n such that every C^2_{n-2} on them is apolar to Q_α. There are $n+2$ such orthic bases in Γ^n. All points of Γ^n whose names contain one symbol form an orthic base, as 12, 13, 14, ... Every R^{n-1} on an orthic base is orthic to Q_α. The following theorems are sufficiently evident from what we have said in discussing the 3- and 4-dimensional cases.

Sets of points $\iota_n^{(n)}$ in α are n-points apolar to Q_α, and inscribed in R^n's of the basic system on any orthic base of Γ^n.

All loci determined in α by basic R^n's are completely determined by Q_α and an orthic base of Γ^n.

Choosing as reference points in α n points of an orthic base of Γ^n, with the $(n+1)$-st point as unit point, a basic R^n on a point x may be given parametrically,

$$y_i=\frac{x_i}{1-x_i t},\quad (i=1,\ \ldots,\ n), \tag{1}$$

the point x having the parameter value $t=0$. We indicate the quadric Q_α by

$$Q=0.$$

The conditions on the coefficients of Q may easily be worked out. The polar S_{n-2} of x as to Q is

$$\Sigma\, y_i \frac{\partial Q}{\partial x_i}=0. \tag{2}$$

(2) cuts out of the R^{n-1} (1) the $n-1$ points which, taken with x, form a set $\iota_n^{(n)}$ in α. The $(n-1)$-ic equation in t giving the parameters of these $n-1$ points may be obtained by substitution of (1) in (2). It is

$$x_1 x_2 \ldots . x_n L t^{n-1} - f_n t^{n-2} + \ldots . + (-)^{n-2} f_3 t + (-)^{n-1} . 2Q = 0, \tag{3}$$

where
$$f_{i+1} \equiv \Sigma x_1 \ldots . x_i \left(\frac{\partial Q}{\partial x_1} + \ldots . + \frac{\partial Q}{\partial x_i} \right),$$
and
$$L \equiv \Sigma \frac{\partial Q}{\partial x_i}.$$

L is the polar S_{n-2} of the unit point as to Q. The spread f_{i+1} is a C_{n-2}^{i+1} on the $C_{n-i-1}^{(i+1)!}$, the locus of points of contact of basic R^n's having $(i+1)$-point contact with α.

The vanishing of an invariant of (3) of degree i and weight w evidently gives as the locus of the point x a C_{n-2}^{w+2i}. Since
$$w = \tfrac{1}{2} i (n-1),$$
we have the theorem:

The locus of points x in α such that the $n-1$ points besides x of the set $\iota_n^{(n)}$ determined by x are $n-1$ points on the basic R^{n-1} on x and an orthic base in Γ^n, and, as points on this R^{n-1} as a binary support, have a vanishing invariant I_i, is a C_{n-2}, of order $\tfrac{1}{2} i (n+3)$.

The n points of a set $\iota_n^{(n)}$ considered as a set of the basic R^n are projectively equivalent to the n points considered as points of the corresponding R^{n-1}, since the R^{n-1} is a projection of the R^n, and these n points are self-corresponding. Hence (3) multiplied by t,
$$x_1 \ldots . x_n L t^n - f_n t^{n-1} + \ldots . = 0, \tag{4}$$
represents to within a collineation on the R^n the n points of intersection with α of the basic R^n on x. The vanishing of an invariant of (4) must, then, carry with it the vanishing of the corresponding invariant of the n points on the basic R^n. An invariant of (4) of degree i and weight w is of degree $w+i$ in the x's. Since $w = \tfrac{1}{2} n i$, we have
$$w + i = \tfrac{1}{2} i (n+2).$$
Hence:

Basic R^n's meeting α in sets $\iota_n^{(n)}$ for which an invariant I_i (on the R^n) vanishes meet α in points of a C_{n-2} of order $\tfrac{1}{2} i (n+2)$.

§ 37. *A Cubic Spread.*

The C_{n-2}'s, $Q_{\alpha+\lambda\beta}$, on S_{n-1}'s of a pencil,
$$(\alpha x) + \lambda (\beta x) = 0,$$
lie on the C_{n-1}^3
$$\begin{vmatrix} (\alpha x^2), & (\beta x^2) \\ (\alpha x), & (\beta x) \end{vmatrix} \equiv \Sigma \pi_{ij} x_i x_j (x_i - x_j) = 0, \tag{5}$$

where $\pi_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$ are the coördinates of the S_{n-2} common to α and β. Now the basic R^n on x passes through x with the tangent $p_{ij} = x_i x_j (x_i - x_j)$. Hence:

The locus of points of tangency with basic R^n's of tangents meeting an S_{n-2}, π, is the C^3_{n-1} (5).

This locus contains the S_{n-2} singly.

Consider two S_{n-2}'s that have an S_{n-3} in common. The two C^3_{n-1}'s determined by the two S_{n-2}'s meet in a C^9_{n-2}, out of which the C^2_{n-2} Q_α on the S_{n-1}, α, common to the two S_{n-2}'s, factors. Hence:

The locus of points of tangency with basic R^n's of tangents meeting an S_{n-3} is a C^7_{n-2}.

Consider an S_{n-2} and an S_{n-3} which have an S_{n-4} in common. There is an S_{n-1}, α, containing both S_{n-2} and S_{n-3}. Tangents to basic R^n's in α are tangents to basic R^{n-1}'s on an orthic base in α at points of Q_α. Lines tangent to basic R^n's and meeting S_{n-3} in α are tangents to basic R^{n-1}'s in α which meet S_{n-3} and have their points of tangency on Q_α. The locus of these points of tangency is the intersection of a C^3_{n-2} with Q_α, — a C^6_{n-3}. The C^3_{n-1} and the C^7_{n-2} determined in S_n by S_{n-2} and S_{n-3} respectively intersect in a C^{21}_{n-3}, out of which the above C^6_{n-3} factors. Hence:

The locus of points of tangency with basic R^n's of tangents meeting an S_{n-4} is a C^{15}_{n-3}.

We can now deduce the general law. Call the order of the C_{n-r+1} of points of tangency with basic R^n's of tangents meeting an S_{n-r}, m_r. Consider an S_{n-2} and an S_{n-r+1} which have an S_{n-r} in common. There is an S_{n-1}, α, containing both S_{n-r+1} and S_{n-2}. For an S_{n-1} containing $n-r+1$ points of S_{n-r}, one further point of S_{n-r+1}, and $r-2$ further points of S_{n-2}, contains both spaces, and is on n points, —enough to fix it. Lines tangent to basic R^n's, in α and meeting S_{n-r+1}, are lines tangent to basic R^{n-1}'s in α and meeting S_{n-r+1}, and having their points of tangency on Q_α. The locus of these points of tangency is the intersection of a C_{n-r+2} of order m_{r-2} with Q_α, —a C_{n-r+1} of order $2m_{r-2}$. The C^3_{n-1} and the C_{n-r+2} of order m_{r-1} determined by S_{n-2} and S_{n-r+1} intersect in a C_{n-r+1} of order $3m_{r-1}$ out of which the above C_{n-r+1} of order $2m_{r-2}$ factors. This gives the recurrence formula

$$m_r = 3m_{r-1} - 2m_{r-2},$$

whence

$$m_r = 2^r - 1.$$

Hence:

The locus of points of tangency with basic R^n's of tangents meeting an S_{n-r} is a C_{n-r+1}, of order 2^r-1.

In particular we have:

The locus of points of tangency with basic R^n's of tangents on a point (S_0) is a C_1, a curve of order 2^n-1.

This curve is projected from a point of it by a cone C_2 of order 2^n-2. Hence:

Lines tangent to basic R^n's and on a point lie on a C_2-cone of order 2^n-2.

§ 38. *A Generalization of the Weddle Surface.*

The argument of § 34, Part IV, obviously holds in n dimensions. We have, then, that the locus, Ψ^n, of points from which $n+3$ points in S_n project into $n+3$ points of an R^{n-1} is a C_2 with these properties:

1) *It has $(n-1)$-fold points at the $n+3$ points.*

2) *It projects from one of these points into a Ψ^{n-1} taken twice.*

If m_n is the order of Ψ^n, we have the recurrence formula

$$m_n = n - 1 + 2\,m_{n-1},$$

whence

$$m_n = 2^n - n - 1.$$

Hence:

The locus of points in S_n from which $n+3$ points project into $n+3$ points of an R^{n-1} is a $C_2^{m_n}$, where m_n has the above value.

This locus has properties closely analogous to those of the Weddle surface.

We have also:

Lines bisecant to basic R^n's and on a point, a, in S_n lie on a C_3 of order m_{n-1}.

The points of intersection with these lines of their bisecant basic R^n's are on the C_2, Ψ^n.

§ 39. *Some Enumerative Theorems.*

We can find the order of the C_{n-2} in which basic R^n's tangent to α meet α again by finding the order of its C_{n-3} of intersection with an S_{n-2} of Γ^n. A basic R^n cannot meet a base S_{n-1} without degenerating, and hence we need consider only the degenerate basic R^n's. And we evidently need consider only the basic R^n's degenerating into lines and R^{n-1}'s, and into conics and R^{n-2}'s. Let us assume that the order of the C_{n-3} of further intersection with an S_{n-2} of tangent

basic R^{n-1}'s in S_{n-1} is m_{n-1}. Basic R^{n-1}'s in the S_{n-1} cutting out the S_{n-2} from Γ^n determine a $C_{n-3}^{m_{n-1}}$ on S_{n-2}. All S_{n-3}'s on the S_{n-2} in Γ^n count doubly in $C_{n-2}^{m_n}$, since two conics of the pencil in the plane opposite to a base S_{n-2} touch α, and either of these conics, taken with a basic R^{n-2} on the opposite S_{n-2}, is an R^n satisfying our conditions. Since there are n S_{n-3}'s on an S_{n-2} of Γ^n, we have the recurrence formula

$$m_n = 2n + m_{n-1},$$

whence

$$m_n = (n+3)(n-2).$$

This gives us the theorem:

Basic R^n's tangent to an S_{n-1}, α, meet α again in a $C_{n-2}^{(n+3)(n-2)}$.

This C_{n-2} is the discriminant of (4). It touches Q_α along the C_{n-3}^6 of points of osculation of basic R^n's with α and cuts out of Q_α the points of contact of bitangent basic R^n's. Hence:

Points of contact with α of bitangent basic R^n's lie on a $C_{n-3}^{2(n+4)(n-3)}$.

It may be proved as in Part IV that lines joining corresponding points of this C_{n-3} are lines of Q_α.

We now ask for the order of the C_{n-p} in which basic R^n's having p-point contact with α meet α again. Call it m_p^n. We can find the order of C_{n-p} by finding the order of the C_{n-p+1} in which it meets an S_{n-2} of Γ^n. The S_{n-p-1}'s of Γ^n count $p!$-fold in the locus, since they are cut out by base S_{n-p}'s, and $p!$ basic R^p's of the systems on the opposite S_p's have p-point contact with α. There are $\binom{n}{n-p+1}$ S_{n-p-1}'s on an S_{n-2} of Γ^n. There are on S_i's of Γ^n, where

$$n-p-1 < i < n-2,$$

spreads belonging to our locus, but their dimension is less than $n-p-1$, and we need not take them into account. On the S_{n-1} cutting out our S_{n-2} we have a basic system of R^{n-1}'s, and these determine a C_{n-p-1}^t on S_{n-2}, where $t = m_p^{n-1}$. Hence we have the recurrence formula

$$m_p^n = \binom{n}{n-p+1} p! + m_p^{n-1},$$

whence

$$m_p^n = p!\left[\binom{n+1}{p} - p - 1\right]. \tag{6}$$

Hence:

Basic R^n's having p-point contact with α meet α again in a C_{n-p}^t, where $t = m_p^n$ as given in (6).

It is $n-i-1$ conditions on a curve in S_n to meet an S_i. If the locus of basic R^n's meeting an S_{n-p} is a C_{n-p+1} of order x_p^n, an S_{p-1} meets this locus in x_p^n points, and there are x_p^n basic R^n's meeting an S_{n-p} and an S_{p-1} by the theorem of Sturm. Hence the locus of basic R^n's meeting an S_{p-1} is met by an S_{n-p} in x_p^n points, and the order of the latter locus is x_p^n. The locus of basic R^n's meeting a C_{n-p}^{μ} is met by an S_{p-1} in the μx_p^n points defined on S_{p-1} by the basic R^n's meeting S_{p-1} and C_{n-p}^{μ}. The order of the locus of basic R^n's meeting C_{n-p}^{μ} is therefore μx_p^n, provided the C_{n-p}^{μ} does not meet a base S_i where $i<p$. We have seen that basic R^n's have p-point contact with α at points of a $C_{n-p}^{p!}$ and meet again in points of a C_{n-p} of order m_p^n. The $C_{n-p}^{p!}$ does not meet a base S_i where $i<p$, since an R^i cannot have p-point contact with α if $i<p$. The locus of basic R^n's meeting $C_{n-p}^{p!}$ is a C_{n-p+1} which meets α in $C_{n-p}^{p!}$ taken p times and in the C_{n-p} of order m_p^n. Hence:

$$x_p^n = \frac{p \cdot p! + m_p^n}{p!} = \binom{n+1}{p} - 1. \tag{7}$$

Therefore:

Basic R^n's meeting an S_{n-p} lie on a C_{n-p+1} of order x_p^n.

There are x_p^n basic R^n's that meet an S_{n-p} and an S_{p-1}.

Basic R^n's meeting an S_{p-1} lie on a C_p of order x_p^n.

The identity involved in these theorems,

$$x_p^n = x_{n-p+1}^n,$$

is evident from the form of (7).

The following theorems are easily verified by the methods of Part IV:

Basic R^n's that touch a C_{n-1}^m meet it again in a C_{n-2}^{μ}, where

$$\mu = m(m+1)\left[\frac{(n-1)(n+2)}{2}m - 2\right].$$

Basic R^n's bitangent to a C_{n-1}^m touch at points of a C_{n-3}^{ν}, where

$$\nu = m(m+1)\left[\frac{(n-1)(n+2)m(m+1)}{2} - 4m - 6\right].$$

The multiplicity of a base S_r in the C_{n-p+1} of basic R^n's meeting an S_{n-p} is $\binom{n-r}{p}$.

By counting of constants we should expect to find a finite number of basic R^n's on a basic C_{n-1}^2. An R^n in S_n is determined by $(n+1)^2-4$ constants. It is $2n+1$ conditions on the R^n to lie on a C_{n-1}^2. It is $n-2$ conditions on a curve lying on a C_{n-1} to pass through a given point of the C_{n-1}. Hence:

An R^n on a C_{n-1}^2 in S_n is determined by

$$\frac{(n+1)^2-4-2n+1}{n-2}=n+2$$

points, — a base, — to within a finite number of positions.

By a theorem of Part I, basic R^n's having $(n-1)$-point contact with a C_{n-1}^2 touch at points of a $C_1^{n!}$. If the C_{n-1}^2 is basic, this $C_1^{n!}$ exists, but can be nothing but basic R^n's, since an R^n having $(n-1)$-point contact with the C_{n-2}^2 has $n+2+n-1=2n+1$ points in common with it, and must lie on it. Hence:

There are $(n-1)!$ basic R^n's on a basic C_{n-1}^2.

Basic R^n's meeting an S_{n-2} r times lie on a C_{n-r}; let us assume that its order is τ_r^n. We obtain a formula for τ_r^n by looking for the C_{n-r-1} in which the C_{n-r} meets a base S_{n-1}. The given S_{n-2} meets a base S_{n-1} in an S_{n-3}; basic R^{n-1}'s on S_{n-1} meeting S_{n-3} r times lie on a C_{n-r-1} of order τ_r^{n-1}. Consider a base S_{n-i} and the opposite S_i. S_{n-2} meets S_i in an S_{i-2}. One basic R^i meets S_{i-2} $i-1$ times. S_{n-2} meets the base S_{n-i} in an S_{n-i-2}. The basic R^{n-i}'s on S_{n-i} meeting this S_{n-i-2} $r-i+1$ times (enough to make S_{n-2} meet R^{n-i} taken with the R^i in the opposite S_i r times) lie on a C_{n-r-1} of order τ_{r-i+1}^{n-i}. There are $\binom{n}{i-1}$ base S_{n-i}'s on a base S_{n-1}. This gives the formula

$$^*\tau_r^n=\tau_r^{n-1}+\binom{n}{1}\tau_{r-1}^{n-2}+\binom{n}{2}\tau_{r-2}^{n-3}+\cdots\cdots+\binom{n}{r}\tau_0^{n-r-1}, \qquad (8)$$

by which τ_r^n may be calculated in any particular case. The following values of special τ's are in accordance with the facts:

$$\tau_0^n=1,\quad \tau_n^n=0\,(n\neq 0),\quad \tau_0^0=1,\quad \tau_n^m=0\,(n>m),\quad \tau_{n-1}^n=n.$$

The following general formula for τ_{n-r}^{n+1} may be verified by induction from (8):

$$\tau_{n-r}^{n+1}=\frac{r!\,(n+1)-(r+1)!\sum_{i=1}^{r}\frac{1}{i}+\sum_{i=1}^{r}(-)^{i+1}(r!)^2(i-1)!\,(r-i+1)^{n+1}}{(r!)^2}.$$

Basic R^n's meeting r S_{n-2}'s lie on a C_{n-r}; let us assume that its order is σ_r^n. An argument similar to the above gives us a formula by means of which σ_r^n may be calculated. Let us examine the C_{n-r-1} in which our C_{n-r} meets a base S_{n-1}. The r S_{n-2}'s meet S_{n-1} in r S_{n-3}'s, and these give a C_{n-r-1} of order σ_r^{n-1} which belongs to our locus. Consider a base S_{n-i} and the opposite S_i. $i-1$ S_{n-2}'s

* It is a remarkable fact that the number τ_{n-1}^{n+1} is the genus of the curve of intersection of $n-1$ C_{n-1}^2's in S_r.

meet S_i in $i-1$ S_{i-2}'s. These can be chosen from the r S_{n-2}'s in $\binom{r}{i-1}$ ways. There are σ^i_{i-1}/i R^i's in S_i meeting these S_{i-2}'s. The remaining $r-i+1$ S_{n-2}'s meet S_{n-i} in $r-i+1$ S_{n-i-2}'s. Basic R^{n-i}'s on S_{n-i} meeting these S_{n-i-2}'s lie on a C_{n-r-1} of order σ^{n-i}_{r-i+1}. There being $\binom{n}{i-1}$ S_{n-i}'s on a base S_{n-1}, we have the formula

$$\sigma^n_r = \sigma^{n-1}_r + \binom{n}{1}\binom{r}{1}\sigma^{n-2}_{r-1}\sigma^2_1/2 + \binom{n}{2}\binom{r}{2}\sigma^{n-3}_{r-2}\sigma^3_2/3 + \dots$$
$$+ \binom{n}{r}\binom{r}{r}\sigma^{n-r-1}_0\sigma^{r+1}_r/r+1. \qquad (9)$$

The following values of special σ's are in accordance with the facts:

$$\sigma^n_0 = 1,\ \ \sigma^n_n = 0\,(n \neq 0),\ \ \sigma^0_0 = 1,\ \ \sigma^m_n = 0\,(n > m),\ \ \sigma^n_{n-1} = n\,\sigma^n_{n-2}.$$

It is easy to calculate any special τ or σ from (8) or (9) taken with the table of special values. Suppose, for example, we require the value of σ^4_3. In the first place,

$$\sigma^4_3 = 4\,\sigma^4_2.$$

From (9) we have

$$\sigma^4_2 = \sigma^3_2 + \binom{4}{1}\binom{2}{1}\sigma^2_1\sigma^2_1/2 + \binom{4}{2}\binom{2}{2}\sigma^1_0\sigma^3_2/3.$$

We have also

$$\sigma^2_1 = 2\,\sigma^2_0 = 2,\ \ \sigma^3_2 = 3\,\sigma^3_1 = 3\left[\sigma^2_1 + \binom{3}{1}\binom{1}{1}\sigma^1_0\sigma^2_1/2\right] = 15,$$

whence

$$\sigma^4_2 = 61,$$

a number which we found in Part III. This gives

$$\sigma^4_3 = 4\,.\,61 = 244.$$

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